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Periodic solutions for a class of perturbed sixth-order autonomous differential equations

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Abstract

Purpose – The objective of this work is to study the periodic solutions for a class of sixth-order autonomous ordinary differential equations $x^{(6)} + (1+p^2+q^2)\ddddot{x} + (p^2+q^2+p^2q^2)\ddot{x} + p^2q^2x = \varepsilon F(x,\dot{x},\ddot{x},\dddot{x},\ddddot{x},x^{(5)})$, where p and q are rational numbers different from $1, 0, -1$ and $p \neq q$, ε is a small enough parameter and $F \in C^2$ is a nonlinear autonomous function.

Design/methodology/approach – The authors shall use the averaging theory to study the periodic solutions for a class of perturbed sixth-order autonomous differential equations (DEs). The averaging theory is a classical tool for the study of the dynamics of nonlinear differential systems with periodic forcing. The averaging theory has a long history that begins with the classical work of Lagrange and Laplace. The averaging theory is used to the study of periodic solutions for second and higher order DEs.

Findings – All the main results for the periodic solutions for a class of perturbed sixth-order autonomous DEs are presenting in the Theorem 1. The authors present some applications to illustrate the main results.

Originality/value – The authors studied Equation 1 which depends explicitly on the independent variable t. Here, the authors studied the autonomous case using a different approach.

Keywords Periodic orbit, Sixth-order differential equation, Averaging theory

Paper type Research paper

1. Introduction

When studying the dynamics of differential systems following the analysis of their equilibrium points, we should study the existence or not of their periodic orbits.

The averaging theory is a classical tool for the study of the dynamics of nonlinear differential systems with periodic forcing. The averaging theory has a long history that begins with the classical work of Lagrange and Laplace. Details of the averaging theory can be found in the books of Verhulst [1] and Sanders and Verhulst [2]. The averaging theory is used to the study of periodic solutions for second and higher order differential equations (DEs) (see Refs [3–7]).

In [8], the authors studied the periodic solution of the following fifth-order differential equation:

$$x^{(5)} - e\ddddot{x} - d\dddot{x} - c\ddot{x} - b\dot{x} - ax = \varepsilon F\left(t,x,\dot{x},\ddot{x},\dddot{x},\ddddot{x}\right), \tag{1}$$

where $a = \lambda\mu\delta$, $b = -(\lambda\mu + \lambda\delta + \mu\delta)$, $c = \lambda + \mu + \delta + \lambda\mu\delta$, $d = -(1 + \lambda\mu + \lambda\delta + \mu\delta)$, $e = \lambda + \mu + \delta$, ε is a small parameter and $F \in C^2$ is $2\pi -$ periodic in t. Here, the variable x and the parameters λ, μ, δ and ε are real.

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In [9], the authors studied equation (1) with $F = F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x})$ which is autonomous. They studied the five cases.

In [10], the authors studied the periodic solution of the following sixth-order differential equation:

$$x^{(6)} + \left(1 + p^2 + q^2\right)\ddddot{x} + \left(p^2 + q^2 + p^2q^2\right)\ddot{x} + p^2q^2x = \varepsilon F\left(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}, x^{(5)}\right), \tag{2}$$

where p and q are rational numbers different from $-1, 0, 1$ and $p \neq q$, ε is small enough real parameter and $F \in C^2$ is a nonlinear nonautonomous periodic function.

Differential equations (DEs are one of the most important tools in mathematical modeling. For examples, the phenomena of physics, fluid and heat flow, motion of objects, vibrations, chemical reactions and nuclear reactions have been modeled by systems of DEs. Many applications of ordinary differential equations (ODEs) of different orders can be found in the mathematical modeling of real-life problems. Second- and third-order DEs can be found in Refs [11–14], and fourth-order DEs often arise in many fields of applied science such as mechanics, quantum chemistry, electronic and control engineering and also beam theory [15], fluid dynamics [16, 17], ship dynamics [18] and neural networks [19]. Numerically and analytically numerous approximations to solve such DEs of various orders have is studied in the literature. Most solutions of the mathematical models of these applications must be approximated.

The objective of this work is to study the periodic solutions for a class of sixth-order autonomous ordinary DEs:

$$x^{(6)} + \left(1 + p^2 + q^2\right)\ddddot{x} + \left(p^2 + q^2 + p^2q^2\right)\ddot{x} + p^2q^2x = \varepsilon F\left(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}, x^{(5)}\right), \tag{3}$$

where p and q are rational numbers different from $-1, 0, 1$, and $p \neq q$, ε is small enough real parameter and $F \in C^2$ is a nonlinear autonomous function.

In general, obtaining analytically periodic solutions of a differential system is a very difficult task, usually impossible. Recently, the study of the periodic solutions of sixth-order of DEs has been considered by several authors (see Refs [3, 20, 21]). Here, using the averaging theory, we reduce this difficult problem for the differential equation (3) to find the zeros of a nonlinear system of five equations. For more information and details about the averaging theory, see section (2) and the references quoted there.

In [10], the authors study the equation (2) where depends explicitly on the independent variable t. Here, we study the autonomous case using a different approach. We shall use the averaging theory to study the periodic solutions for a class of sixth-order autonomous differential equation (3).

Now, all our main results for the periodic solutions of equation (3) are as follows:

Theorem 1. *Assume that p, q are rational numbers different from 1, 0, -1 and $p \neq q$, in DE* (3). *For every positive simple $(r_0^*, Z_0^*, U_0^*, V_0^*, W_0^*)$ solution of the system,*

$$\mathcal{F}_i(r_0, Z_0, U_0, V_0, W_0) = 0, \quad i = 1, \ldots, 5, \tag{4}$$

satisfying

$$\det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5)}{\partial(r_0, Z_0, U_0, V_0, W_0)}\Big|_{(r_0, Z_0, U_0, V_0, W_0) = \left(r_0^*, Z_0^*, U_0^*, V_0^*, W_0^*\right)}\right) \neq 0, \tag{5}$$

where

$$\mathcal{F}_1(r_0,Z_0,U_0,V_0,W_0)=\frac{1}{2\pi k}\int_0^{2\pi k}\cos\theta F(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4,\mathcal{A}_5,\mathcal{A}_6)d\theta,$$
$$\mathcal{F}_2(r_0,Z_0,U_0,V_0,W_0)=\frac{1}{2\pi k}\int_0^{2\pi k}\frac{-pU_0\sin\theta+r_0\cos(p\theta)}{r_0}F(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4,\mathcal{A}_5,\mathcal{A}_6)d\theta,$$
$$\mathcal{F}_3(r_0,Z_0,U_0,V_0,W_0)=\frac{1}{2\pi k}\int_0^{2\pi k}\frac{pZ_0\sin\theta-r_0\sin(p\theta)}{r_0}F(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4,\mathcal{A}_5,\mathcal{A}_6)d\theta,$$
$$\mathcal{F}_4(r_0,Z_0,U_0,V_0,W_0)=\frac{1}{2\pi k}\int_0^{2\pi k}\frac{-qW_0\sin\theta+r_0\cos(q\theta)}{r_0}F(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4,\mathcal{A}_5,\mathcal{A}_6)d\theta,$$
$$\mathcal{F}_5(r_0,Z_0,U_0,V_0,W_0)=\frac{1}{2\pi k}\int_0^{2\pi k}\frac{qV_0\sin\theta-r_0\sin(q\theta)}{r_0}F(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4,\mathcal{A}_5,\mathcal{A}_6)d\theta,$$
(6)

be with $p = p_1/p_2$, $q = q_1/q_2$, where p_1, p_2, q_1, q_2 are positive integers $p \neq q, (p_1, p_2) = (q_1, q_2) = 1$, let k be the least common multiple of p_2 and q_2, and

$$\mathcal{A}_1=\frac{r_0\sin\theta}{(q^2-1)(p^2-1)}+\frac{U_0\cos(p\theta)+Z_0\sin(p\theta)}{p(p^2-q^2)(p^2-1)}-\frac{W_0\cos(q\theta)+V_0\sin(q\theta)}{q(p^2-q^2)(q^2-1)},$$
$$\mathcal{A}_2=\frac{r_0\cos\theta}{(q^2-1)(p^2-1)}+\frac{Z_0\cos(p\theta)-U_0\sin(p\theta)}{(p^2-q^2)(p^2-1)}+\frac{-V_0\cos(q\theta)+W_0\sin(q\theta)}{(p^2-q^2)(q^2-1)},$$
$$\mathcal{A}_3=-\frac{r_0\sin\theta}{(q^2-1)(p^2-1)}-\frac{p(U_0\cos(p\theta)+Z_0\sin(p\theta))}{(p^2-q^2)(p^2-1)}+\frac{q(W_0\cos(q\theta)+V_0\sin(q\theta))}{(p^2-q^2)(q^2-1)},$$
$$\mathcal{A}_4=-\frac{r_0\cos\theta}{(q^2-1)(p^2-1)}+\frac{p^2(-Z_0\cos(p\theta)+U_0\sin(p\theta))}{(p^2-q^2)(p^2-1)}+\frac{q^2(V_0\cos(q\theta)-W_0\sin(q\theta))}{(p^2-q^2)(q^2-1)},$$
$$\mathcal{A}_5=-\frac{r_0\cos\theta}{(q^2-1)(p^2-1)}+\frac{p^3(U_0\cos(p\theta)+Z_0\sin(p\theta))}{(p^2-q^2)(p^2-1)}-\frac{q^3(W_0\cos(q\theta)+V_0\sin(q\theta))}{(p^2-q^2)(q^2-1)},$$
$$\mathcal{A}_6=\frac{r_0\sin\theta}{(q^2-1)(p^2-1)}+\frac{p^4(Z_0\cos(p\theta)-U_0\sin(p\theta))}{(p^2-q^2)(p^2-1)}+\frac{q^4(-V_0\cos(q\theta)+W_0\sin(q\theta))}{(p^2-q^2)(q^2-1)},$$
(7)

There is a periodic solution $x(t,\varepsilon)$ of equation (3) *tending to the periodic solution*

$$x(t)=\frac{r_0^*\sin t}{(q^2-1)(p^2-1)}+\frac{U_0^*\cos(pt)+Z_0^*\sin(pt)}{p(p^2-q^2)(p^2-1)}-\frac{W_0^*\cos(qt)+V_0^*\sin(qt)}{q(p^2-q^2)(q^2-1)}, \qquad (8)$$

of the equation $x^{(6)}+(1+p^2+q^2)\ddddot{x}+(p^2+q^2+p^2q^2)\ddot{x}+p^2q^2x=0$, when $\varepsilon\to 0$. Note that this solution is periodic of period $2\pi k$.

Theorem 1 is proved in section 3. Two applications of Theorem 1 are as follows:

Corollary 2. *If $F(x,\dot{x},\ddot{x},\dddot{x},\ddddot{x},x^{(5)})=\dot{x}^2+\ddot{x}^2-\dddot{x}$, then the differential* equation (3) *with $p=2, q=\frac{1}{2}$ has four periodic solutions $x_i(t,\varepsilon)$ for $i=1,\ldots,4$ tending to the periodic solutions*

$$\begin{aligned} x_1(t) &= -\frac{1}{6}sint + \frac{4}{3}\sin\left(\frac{1}{2}t\right), \\ x_2(t) &= -\frac{1}{6}sint - \frac{4}{3}\sin\left(\frac{1}{2}t\right), \\ x_3(t) &= \frac{1}{6}sint + \frac{4}{3}\cos\left(\frac{1}{2}t\right), \\ x_4(t) &= \frac{1}{6}sint - \frac{4}{3}\cos\left(\frac{1}{2}t\right), \end{aligned}$$

of $x^{(6)} + \frac{21}{4}\ddddot{x} + \frac{21}{4}\ddot{x} + x = 0$ *when* $\varepsilon \to 0$.

Corollary 2 is proved in section 5.

Corollary 3. *If* $F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}, x^{(5)}) = -\dot{x}^2 - 2\dot{x}$, *then the differential equation (3) with* $p = 2$, $q = 3$ *has four periodic solutions* $x_i(t, \varepsilon)$ *for* $i = 1, \ldots, 4$ *tending to the periodic solutions*

$$\begin{aligned} x_1(t) &= \frac{2}{5}\sqrt{25-5\sqrt{5}}\sin t - \frac{1}{30}\left(15-15\sqrt{5}\right)\sin(2t) - \frac{1}{120}\left(8-8\sqrt{5}\right)\sqrt{25-5\sqrt{5}}\sin(3t), \\ x_2(t) &= -\frac{2}{5}\sqrt{25-5\sqrt{5}}\sin t - \frac{1}{30}\left(15-15\sqrt{5}\right)\sin(2t) + \frac{1}{120}\left(8-8\sqrt{5}\right)\sqrt{25-5\sqrt{5}}\sin(3t), \\ x_3(t) &= \frac{2}{5}\sqrt{25+5\sqrt{5}}\sin t - \frac{1}{30}\left(15+15\sqrt{5}\right)\sin(2t) - \frac{1}{120}\left(8+8\sqrt{5}\right)\sqrt{25+5\sqrt{5}}\sin(3t), \\ x_4(t) &= -\frac{2}{5}\sqrt{25+5\sqrt{5}}\sin t - \frac{1}{30}\left(15+15\sqrt{5}\right)\sin(2t) + \frac{1}{120}\left(8+8\sqrt{5}\right)\sqrt{25+5\sqrt{5}}\sin(3t), \end{aligned}$$

of $x^{(6)} + 14\ddddot{x} + 49\ddot{x} + 36x = 0$ *when* $\varepsilon \to 0$.

Corollary 3 is proved in section 5.

2. Averaging theory

In this section, we present the basic results from the averaging theory that we shall need for proving the main results of this paper. We want to study the T-periodic solutions of the periodic differential systems of the form

$$\dot{\mathbf{x}} = F_0(\mathbf{x}, t) + \varepsilon F_1(\mathbf{x}, t) + \varepsilon^2 F_2(\mathbf{x}, t, \varepsilon), \tag{9}$$

with $\varepsilon > 0$ sufficiently small. The functions $F_0, F_1 : \Omega \times \mathbb{R} \to \mathbb{R}^n$ and $F_2 : \Omega \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are C^2 functions, T-periodic in the variable t and Ω is an open subset of $\mathbb{R}^n$. We denote by $\mathbf{x}(\mathbf{z}, t, \varepsilon)$ the solution of the differential system (9) such that $\mathbf{x}(\mathbf{z}, 0, \varepsilon) = \mathbf{z}$. We assume that the unperturbed system

$$\dot{\mathbf{x}} = F_0(\mathbf{x}, t), \tag{10}$$

has an open set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, $\mathbf{x}(t, \mathbf{z}, 0)$ is T-periodic.

We consider the variational equation

$$\dot{\mathbf{y}} = D_{\mathbf{x}}F_0(\mathbf{x}(\mathbf{z}, t, 0), t)\mathbf{y}, \tag{11}$$

of the unperturbed system on the periodic solution $\mathbf{x}(\mathbf{z}, t, 0)$, where y is an $n \times n$ matrix. Let $M_{\mathbf{z}}(t)$ be the fundamental matrix of the linear differential system (11) such that $M_{\mathbf{z}}(0)$ is the $n \times n$ identity matrix. The next result is due to Malkin [22] and Roseau [23], for a shorter and easier proof see Ref. [24].

Theorem 4. ***[Perturbations of an isochronous set]*** *Consider the function* $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^n$

$$\mathcal{F}(\mathbf{z}) = \int_0^T M_{\mathbf{z}}^{-1}(\mathbf{z}, t) F_1(\mathbf{x}(\mathbf{z}, t), t) dt. \tag{12}$$

If there exists $a \in V$ *with* $\mathcal{F}(a) = 0$ *and* $\det((d\mathcal{F}/d\mathbf{z})(a)) \neq 0$*, then there exists a* T*-periodic solution of system* (9) *such that when* $\varepsilon \to 0$ *we have that* $\mathbf{x}(0, \varepsilon) \to a$.

3. Proof of Theorem 1

If $y = \dot{x}$, $z = \ddot{x}$, $u = \dddot{x}$, $v = \ddddot{x}$, $w = \overset{\cdot\cdot\cdot\cdot\cdot}{x}$, then system (3) can be written as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = u, \\ \dot{u} = v, \\ \dot{v} = w, \\ \dot{w} = -p^2q^2x - \left(p^2 + q^2 + p^2q^2\right)z - \left(1 + p^2 + q^2\right)v + \varepsilon F(x, y, z, u, v, w), \end{cases} \tag{13}$$

with $\varepsilon = 0$, system (13) has a unique singular point at the origin. The eigenvalues of the linear part of this system are $\pm i$, $\pm pi$ and $\pm qi$. By the linear inversible transformation,

$$(X, Y, Z, U, V, W)^T = B(x, y, z, u, v, w)^T, \tag{14}$$

where

$$B = \begin{pmatrix} 0 & p^2q^2 & 0 & p^2 + q^2 & 0 & 1 \\ p^2q^2 & 0 & p^2 + q^2 & 0 & 1 & 0 \\ 0 & q^2 & 0 & q^2 + 1 & 0 & 1 \\ q^2p & 0 & p(q^2 + 1) & 0 & p & 0 \\ 0 & p^2 & 0 & 1 + p^2 & 0 & 1 \\ p^2q & 0 & q(1 + p^2) & 0 & q & 0 \end{pmatrix},$$

We obtain the transformation of the system (13) as follows:

$$\begin{cases} \dot{X} = -Y + \varepsilon G(X, Y, Z, U, V, W), \\ \dot{Y} = X, \\ \dot{Z} = -pU + \varepsilon G(X, Y, Z, U, V, W)), \\ \dot{U} = pZ, \\ \dot{V} = -qW + \varepsilon G(X, Y, Z, U, V, W), \\ \dot{W} = qV, \end{cases} \tag{15}$$

where $G(X, Y, Z, U, V, W) = F(A, B, C, D, J, L)$ with

$$A=\frac{q(q^2-1)U-p(p^2-1)W+pq(p^2-q^2)Y}{pq(p^2-1)(q^2-1)(p^2-q^2)},$$
$$B=\frac{-(p^2-1)V+(p^2-q^2)X+(q^2-1)Z}{(p^2-1)(q^2-1)(p^2-q^2)},$$
$$C=\frac{-p(q^2-1)U+q(p^2-1)W-(p^2-q^2)Y}{(p^2-1)(q^2-1)(p^2-q^2)},$$
$$D=\frac{q^2(p^2-1)V-(p^2-q^2)X-p^2(q^2-1)Z}{(p^2-1)(q^2-1)(p^2-q^2)},$$
$$J=\frac{p^3(q^2-1)U-q^3(p^2-1)W+(p^2-q^2)Y}{(p^2-1)(q^2-1)(p^2-q^2)},$$
$$L=\frac{-q^4(p^2-1)V+(p^2-q^2)X+p^4(q^2-1)Z}{(p^2-1)(q^2-1)(p^2-q^2)}.$$

The linear part of the system (15) at the origin is in its real Jordan normal form and that the change of variables (14) is defined when p and q are different from $1, 0, -1$ and $p \neq q$ because the determinant of the matrix of the change is $-pq(p^2-1)^2(q^2-1)^2(p^2-q^2)^2$. We pass from the cartesian variables (X, Y, Z, U, V, W) to the cylindrical variables (r, θ, Z, U, V, W) of $\mathbb{R}^6$, where $X = r\cos\theta$ and $Y = r\sin\theta$. In these new variables, the differential system (15) can be written as

$$\begin{cases}\dot{r}=\varepsilon\cos\theta H(r,\theta,Z,U,V,W),\\ \dot{\theta}=1-\varepsilon\dfrac{\sin\theta}{r}H(r,\theta,Z,U,V,W),\\ \dot{Z}=-pU+\varepsilon H(r,\theta,Z,U,V,W),\\ \dot{U}=pZ,\\ \dot{V}=-qW+\varepsilon H(r,\theta,Z,U,V,W),\\ \dot{W}=qV,\end{cases}\tag{16}$$

where $H(r, \theta, Z, U, V, W) = F(a, b, c, d, j, l)$ with

$$a=\frac{q(q^2-1)U-p(p^2-1)W+pq(p^2-q^2)r\sin\theta}{pq(p^2-1)(q^2-1)(p^2-q^2)},$$
$$b=\frac{-(p^2-1)V+(p^2-q^2)r\cos\theta+(q^2-1)Z}{(p^2-1)(q^2-1)(p^2-q^2)},$$
$$c=\frac{-p(q^2-1)U+q(p^2-1)W-(p^2-q^2)r\sin\theta}{(p^2-1)(q^2-1)(p^2-q^2)},$$

$$d = \frac{q^2(p^2-1)V - (p^2-q^2)r\cos\theta - p^2(q^2-1)Z}{(p^2-1)(q^2-1)(p^2-q^2)},$$
$$j = \frac{p^3(q^2-1)U - q^3(p^2-1)W + (p^2-q^2)r\sin\theta}{(p^2-1)(q^2-1)(p^2-q^2)},$$
$$l = \frac{-q^4(p^2-1)V + (p^2-q^2)r\cos\theta + p^4(q^2-1)Z}{(p^2-1)(q^2-1)(p^2-q^2).}.$$

Dividing by $\dot{\theta}$, the system (16) becomes

$$\begin{cases} \dfrac{dr}{d\theta} = \varepsilon\cos\theta H + O(\varepsilon^2), \\ \dfrac{dZ}{d\theta} = -pU + \varepsilon\left(1 - \dfrac{pU\sin\theta}{r}\right)H + O(\varepsilon^2), \\ \dfrac{dU}{d\theta} = pZ + \varepsilon\dfrac{pZ\sin\theta}{r}H + O(\varepsilon^2), \\ \dfrac{dV}{d\theta} = -qW + \varepsilon\left(1 - \dfrac{qW\sin\theta}{r}\right)H + O(\varepsilon^2), \\ \dfrac{dW}{d\theta} = qV + \varepsilon\dfrac{qV\sin\theta}{r}H + O(\varepsilon^2), \end{cases} \tag{17}$$

where $H = H(r, \theta, Z, U, V, W)$.

We will now apply Theorem 4 to the system (17). We note that system (17) can be written as system (9) taking

$$x = \begin{pmatrix} r \\ Z \\ U \\ V \\ W \end{pmatrix}, t = \theta,\ F_0(\theta, x) = \begin{pmatrix} 0 \\ -pU \\ pZ \\ -qW \\ qV \end{pmatrix},\ F_1(\theta, x) = \begin{pmatrix} \cos\theta H \\ \left(1 - \dfrac{pU\sin\theta}{r}\right)H \\ \dfrac{pZ\sin\theta}{r}H \\ \left(1 - \dfrac{qW\sin\theta}{r}\right)H \\ \dfrac{qV\sin\theta}{r}H \end{pmatrix}.$$

System (17) with $\varepsilon = 0$ has the $2\pi k$ periodic solutions

$$\begin{pmatrix} r(\theta) \\ Z(\theta) \\ U(\theta) \\ V(\theta) \\ W(\theta) \end{pmatrix} = \begin{pmatrix} r_0 \\ Z_0\cos(p\theta) - U_0\sin(p\theta) \\ U_0\cos(p\theta) + Z_0\sin(p\theta) \\ V_0\cos(q\theta) - W_0\sin(q\theta) \\ W_0\cos(q\theta) + V_0\sin(q\theta) \end{pmatrix},$$

for (r_0, Z_0, U_0, V_0, W_0) with $r_0 > 0$ and $p = p_1/p_2$, $q = q_1/q_2$, where p_1, p_2, q_1, q_2 are positive integers $p \neq q$, $(p_1, p_2) = (q_1, q_2) = 1$, let k be the least common multiple of p_2 and q_2. To look for the periodic solutions of our equation (17), we must calculate the zeros $\alpha = (r_0, Z_0, U_0, V_0, W_0)$ of the system $\mathcal{F}(\alpha) = 0$, where $\mathcal{F}(\alpha)$ is given (12). The fundamental matrix $M(\theta)$ of the system (17) with $\varepsilon = 0$ along any periodic solution is

$$M(\theta) = M_{z_\alpha}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos(p\theta) & -\sin(p\theta) & 0 & 0 \\ 0 & \sin(p\theta) & \cos(p\theta) & 0 & 0 \\ 0 & 0 & 0 & \cos(q\theta) & -\sin(q\theta) \\ 0 & 0 & 0 & \sin(q\theta) & \cos(q\theta) \end{pmatrix}.$$

Now computing the function $\mathcal{F}(\alpha)$ is given (12), we got that the system $\mathcal{F}(\alpha) = 0$ can be written as

$$\begin{pmatrix} \mathcal{F}_1(r, Z, U, V, W) \\ \mathcal{F}_2(r, Z, U, V, W) \\ \mathcal{F}_3(r, Z, U, V, W) \\ \mathcal{F}_4(r, Z, U, V, W) \\ \mathcal{F}_5(r, Z, U, V, W) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{18}$$

where

$$\mathcal{F}_1(r, Z, U, V, W) = \frac{1}{2\pi k}\int_0^{2\pi k} \cos\theta F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6)d\theta,$$

$$\mathcal{F}_2(r, Z, U, V, W) = \frac{1}{2\pi k}\int_0^{2\pi k} \frac{-pU_0\sin\theta + r_0\cos(p\theta)}{r_0} F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6)d\theta,$$

$$\mathcal{F}_3(r, Z, U, V, W) = \frac{1}{2\pi k}\int_0^{2\pi k} \frac{pZ_0\sin\theta - r_0\sin(p\theta)}{r_0} F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6)d\theta,$$

$$\mathcal{F}_4(r, Z, U, V, W) = \frac{1}{2\pi k}\int_0^{2\pi k} \frac{-qW_0\sin\theta + r_0\cos(q\theta)}{r_0} F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6)d\theta,$$

$$\mathcal{F}_5(r, Z, U, V, W) = \frac{1}{2\pi k}\int_0^{2\pi k} \frac{qV_0\sin\theta - r_0\sin(q\theta)}{r_0} F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6)d\theta,$$

with $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5$, and $\mathcal{A}_6$ as in the statement of Theorem 1.

If determinant (5) is nonzero, the zeros $(r^*, Z^*, U^*, V^*, W^*)$ of system (18) with respect to the variable r, Z, U, V and W providing periodic orbits of system (17) with $\varepsilon \neq 0$ small enough if they are simple. Going back to the change of variable, for all simple zero $(r^*, Z^*, U^*, V^*, W^*)$ of system (18), we obtain a $2\pi k$ periodic solution $x(t)$ of the differential equation (3) for $\varepsilon \neq 0$ small enough such that

$$x(t, \varepsilon) \to \frac{r_0^*\sin t}{(q^2-1)(p^2-1)} + \frac{U_0^*\cos(pt) + Z_0^*\sin(pt)}{p(p^2-q^2)(p^2-1)} - \frac{W_0^*\cos(qt) + V_0^*\sin(qt)}{q(p^2-q^2)(q^2-1)},$$

of $x^{(6)} + (1 + p^2 + q^2)\ddddot{x} + (p^2 + q^2 + p^2q^2)\ddot{x} + p^2q^2x = 0$ when $\varepsilon \to 0$, where k is defined in the statement of Theorem 1. Note that this solution is periodic of period $2\pi k$. Theorem 1 is proved.

4. Proof of Corollaries 2 and 3

4.1 Proof of Corollary 2

Consider the function

$$F\left(x,\dot{x},\ddot{x},\dddot{x},\ddddot{x},x^{(5)}\right)=\dot{x}^2+\ddot{x}^2-\dddot{x},$$

which corresponds to the case $p=2$ and $q=\frac{1}{2}$. The functions $\mathcal{F}_i=\mathcal{F}_i(r_0,Z_0,U_0,V_0,W_0)$ for $i=1,\ldots,5$ of Theorem 1 are

$$\begin{aligned}
\mathcal{F}_1 &= \frac{16}{675}\left(V_0^2-W_0^2\right)-\frac{8}{135}r_0Z_0-\frac{2}{9}r_0,\\
\mathcal{F}_2 &= -\frac{8}{675}\frac{10U_0^2r_0-8U_0V_0W_0-15Z_0r_0}{r_0},\\
\mathcal{F}_3 &= -\frac{8}{675}\frac{10U_0Z_0r_0-8Z_0V_0W_0+15U_0r_0}{r_0},\\
\mathcal{F}_4 &= -\frac{2}{675}\frac{10U_0W_0r_0-8V_0W_0^2+40V_0r_0^2-15V_0r_0}{r_0},\\
\mathcal{F}_5 &= \frac{2}{675}\frac{10U_0V_0r_0-8V_0^2W_0+40W_0r_0^2+15W_0r_0}{r_0}.
\end{aligned}$$

System $\mathcal{F}_1=\mathcal{F}_2=\mathcal{F}_3=\mathcal{F}_4=\mathcal{F}_5=0$ has the four solutions:

$$\begin{aligned}
\left(r_0^*,Z_0^*,U_0^*,V_0^*,W_0^*\right) &= \left(\frac{3}{8},0,0,\frac{15}{8},0\right),\\
\left(r_0^*,Z_0^*,U_0^*,V_0^*,W_0^*\right) &= \left(-\frac{3}{8},0,0,\frac{15}{8},0\right),\\
\left(r_0^*,Z_0^*,U_0^*,V_0^*,W_0^*\right) &= \left(-\frac{3}{8},0,0,0,\frac{15}{8}\right),\\
\left(r_0^*,Z_0^*,U_0^*,V_0^*,W_0^*\right) &= \left(-\frac{3}{8},0,0,0,-\frac{15}{8}\right).
\end{aligned}$$

Since the Jacobian

$$\det\left(\frac{\partial(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3,\mathcal{F}_4,\mathcal{F}_5)}{\partial(r_0,Z_0,U_0,V_0,W_0)}\Big|_{(r_0,Z_0,U_0,V_0,W_0)=\left(r_0^*,Z_0^*,U_0^*,V_0^*,W_0^*\right)}\right)=\frac{1024}{12301875}\neq 0$$

by Theorem 1 equation (3) has the four periodic solution of the statement of the Corollary 2. □

4.2 Proof of Corollary 3

Consider the function

$$F\left(x,\dot{x},\ddot{x},\dddot{x},\ddddot{x},x^{(5)}\right)=-\dot{x}^2-2\dot{x},$$

which corresponds to the case $p=2$ and $q=3$. The functions $\mathcal{F}_i=\mathcal{F}_i(r_0, Z_0, U_0, V_0, W_0)$ for $i=1,\ldots,5$ of Theorem 1 are

$$
\begin{aligned}
\mathcal{F}_1 &= -\frac{1}{1200}(W_0U_0+V_0Z_0)-\frac{1}{720}r_0Z_0-\frac{1}{24}r_0,\\
\mathcal{F}_2 &= \frac{1}{57600}\,\frac{96U_0^2V_0-160U_0^2r_0-96U_0W_0Z_0+30V_0r_0^2+25r_0^3+3840Z_0r_0}{r_0},\\
\mathcal{F}_3 &= -\frac{1}{28800}\,\frac{48U_0V_0Z_0-80U_0Z_0r_0-48W_0Z_0^2-15W_0r_0^2-1920U_0r_0}{r_0},\\
\mathcal{F}_4 &= \frac{1}{3600}\,\frac{9U_0V_0W_0-15U_0W_0r_0-9W_0^2Z_0-5Z_0r_0^2-90V_0r_0}{r_0},\\
\mathcal{F}_5 &= -\frac{1}{3600}\,\frac{9U_0V_0^2-15U_0V_0r_0+5U_0r_0-9V_0W_0Z_0+90W_0r_0}{r_0}.
\end{aligned}
$$

System $\mathcal{F}_1=\mathcal{F}_2=\mathcal{F}_3=\mathcal{F}_4=\mathcal{F}_5=0$ has the four solutions:

$$
(r_0^*,Z_0^*,U_0^*,V_0^*,W_0^*)=\left(\frac{48}{5}\sqrt{25-5\sqrt{5}},15-15\sqrt{5},0,\left(-8+8\sqrt{5}\right)\sqrt{25-5\sqrt{5}},0\right),
$$

$$
(r_0^*,Z_0^*,U_0^*,V_0^*,W_0^*)=\left(-\frac{48}{5}\sqrt{25-5\sqrt{5}},15-15\sqrt{5},0,\left(8-8\sqrt{5}\right)\sqrt{25-5\sqrt{5}},0\right),
$$

$$
(r_0^*,Z_0^*,U_0^*,V_0^*,W_0^*)=\left(\frac{48}{5}\sqrt{25+5\sqrt{5}},15+15\sqrt{5},0,-\left(8+8\sqrt{5}\right)\sqrt{25+5\sqrt{5}},0\right),
$$

$$
(r_0^*,Z_0^*,U_0^*,V_0^*,W_0^*)=\left(-\frac{48}{5}\sqrt{25+5\sqrt{5}},15+15\sqrt{5},0,\left(8+8\sqrt{5}\right)\sqrt{25+5\sqrt{5}},0\right).
$$

Since the Jacobian (5) for these four solutions $(r_0^*, Z_0^*, U_0^*, V_0^*, W_0^*)$ is

$$
-\frac{73}{11520000}+\frac{109}{34560000}\sqrt{5},\quad -\frac{73}{11520000}-\frac{109}{34560000}\sqrt{5},
$$

respectively, we obtain using Theorem 1, the four solutions given in statement of the Corollary 3. □

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