

On subcopula estimation for discrete models

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Abstract

Purpose – To discuss subcopula estimation for discrete models.

Design/methodology/approach – The convergence of estimators is considered under the weak convergence of distribution functions and its equivalent properties known in prior works.

Findings – The domain of the true subcopula associated with discrete random variables is found to be discrete on the interior of the unit hypercube. The construction of an estimator in which their domains have the same form as that of the true subcopula is provided, in case, the marginal distributions are binomial.

Originality/value – To the best of our knowledge, this is the first time such an estimator is defined and proved to be converged to the true subcopula.

Keywords Copula, Discrete model, Empirical subcopula, Subcopula

Paper type Research paper

1. Introduction

In the last decade, copulas have been successfully used in many multivariate models appearing in various fields such as economic, finance, agriculture, hydrology, etc. One of the reasons is that copula models allow us to investigate the behavior of each random variable separately before combining their behavior via Sklar's theorem.

According to Sklar's theorem, any joint distribution function H with marginals H_1, \dots, H_k can be written as:

$$H(\vec{x}) = C(H_1(x_1), \dots, H_k(x_k)) \quad (1)$$

for all $\vec{x} \in \mathbb{R}^k$ where C is a copula. Moreover, the converse is also true – any function H defined as in equation (1) for some copula C will always be a joint distribution function with marginals H_1, \dots, H_k . As a result, copulas can be seen as functions that link marginal distribution functions together. (copula means link or connection in Latin). They are functions that contain dependence structures among random variables. This leads to, for



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example, the requirement that a measure of association should be able to be written in terms of copulas in order to remove the effect of marginal distributions, which is also known as scale-free property. This idea has been carried through several measures of association such as Spearman's rank correlation coefficient, Hoeffding's Phi-square, and several measures of functional dependence (Siburg and Stoimenov, 2010; Dette *et al.*, 2013; Tasena and Dhompogsa, 2013, 2016; Boonmee and Tasena, 2016). (see also (Tasena, 2020) for a recent survey.)

At first glance, the above arguments are reasonable. If we were to carefully investigate equation (1), however, we will see that the values of the copula C that effect H are only those that lie on $\prod_{i=1}^k \text{Range}(H_i)$. Therefore, any two different copulas that agree on the set $\prod_{i=1}^k \text{Range}(H_i)$ will define the same joint distribution function H . This might not pose a problem if a copula were only used to define H , but this will not be the case if we need to extract dependence structure. Consider the problem of defining measure of association again. If there are more than one copulas that can be used to define the distribution function H , then which one should be used to compute the association level among random variables. After all, different choices might lead to different values yielding inconsistent results. This is referred to as an identification problem: if we need to identify the copula responsible for the dependence structure among random variables, which one should be used?

In the past, identification is never an issue since copulas are only used in continuous models. If the joint distribution function H is continuous, so are its associated marginal distribution functions H_1, \dots, H_k . Thus, the range of H_i always contains the open unit interval $(0, 1)$ for all $i = 1, \dots, k$. Therefore, all copulas associated to H must at least agree on $(0, 1)^k$. Since any copula is continuous, we can extend this agreement to the unit hypercube. Therefore, we can conclude that there is only one such copula.

The situation is different when one or more random variables are discrete. Say, for example, the first marginal H_1 is a Bernoulli distribution function. Then $\text{Range}(H_1)$ is a three-point set that is rather small comparing to the unit interval. As a result, there are usually infinitely many copulas that can be used to define the joint distribution function H . See de Amo *et al.* (2017) for characterization of such copulas. This situation appears in all discrete models. We do not suggest that using copula extension is always a bad idea. We simply state that there is usually no justification for using one particular form of extension over another. Even if there is, it will only apply to a very specific situation. Also, several discrete models include the form of marginal distribution functions. Therefore, the form of the domains of subcopulas is also known as well. For example, if we know that all marginals are Bernoulli, then the domain must be a product of three-point sets. If all marginals are Binomial, then the domain must be a product of sets of the form:

$$\{0\} \cup \left\{ \sum_{i=0}^k \binom{m}{i} p^i (1-p)^{m-i} \mid k = 0, \dots, m \right\}$$

with the usual parameters p and m . The same applies to Poisson distribution functions, etc. We should not ignore this information when constructing an estimator, should we?

So instead of focusing on the whole unit hypercube $[0, 1]^k$, we suggest focusing only on the set $\prod_{i=1}^k \text{Range}(H_i)$ since this is all the information we can infer for H in equation (1). In other words, we should simply be focusing on a subcopula obtained by restricting the copula C on $\prod_{i=1}^k \text{Range}(H_i)$ instead of the copula C itself.

Mathematically, a subcopula is simply a restriction of a copula to a closed set of the form $A = \prod_{i=1}^k A_i$ with the condition that $\{0, 1\}^k \subseteq A$. In the case of [equation \(1\)](#), the subcopula associated with the joint distribution function H is obtained by restricted the domain of C to $\prod_{i=1}^k \text{Range}(H_i)$. Since all copulas are continuous, it can be proved that two copulas agree on $\prod_{i=1}^k \text{Range}(H_i)$ if and only if they are agree on $\prod_{i=1}^k \text{Range}(H_i)$. Therefore, the subcopula associated with a joint distribution function is unique. Thus, the identification issue is resolved.

It should be mentioned that using subcopulas also brings another complication to model estimations. As we already know, the true joint distribution function is always unknown and has to be estimated from the data. Therefore, we will also have to estimate the true subcopula from data. Since the domain of the true subcopula depended on the marginal distribution functions, it is also unknown and has to be also estimated. (This situation never appears in continuous models since the domains of copulas are always known.) Using plugin estimators will only partially solve the problem. Say, for example, we estimate the marginal distribution functions H_i with its empirical version H_{in} . Then we may estimate $\prod_{i=1}^k \text{Range}(H_i)$ with $\prod_{i=1}^k \text{Range}(H_{in}) = \prod_{i=1}^k \text{Range}(H_{in})$. Similarly, we may construct the empirical subcopula S_n from the empirical distribution function H_n of H . How can we justify whether S_n is a good estimator of the true subcopula S ? Recall that the domain of S is $\prod_{i=1}^k \text{Range}(H_i)$ while the domain of S_n will be $\prod_{i=1}^k \text{Range}(H_{in})$ that is clearly varied with n . So we will have to compare functions with different domains. Something that cannot be done directly. This is probably one of the reasons why subcopula estimation is harder than copula estimation.

Recently, little work has been done to resolve this issue. The basic idea is to embed the set of subcopulas into another set in which the concept of convergence is defined. In other words, we need to replace a subcopula S with its representation, say, $r(S)$ so that we may define $S_n \rightarrow S$ as $r(S_n) \rightarrow r(S)$. [de Amo et al. \(2017\)](#) is the first one who worked in this direction where they represent a subcopula S with its graph so that we have $S_n \rightarrow S$ if and only if their corresponding graphs converge under the Hausdorff distance in this case.

[Rachasingho and Tasena \(2020\)](#), on the other hand, identify a subcopula with the class of its copula extensions. This provides a relationship between the convergence of subcopulas and their corresponding copula extensions. In order to resolve the identification problem, they also suggested that the distribution forms of subcopulas be used instead ([Rachasingho and Tasena, 2018; Tasena, 2021a, b](#)). A nice property of distribution forms is that they do not change the support of the underlying measures. Hence, they will never affect the dependence structure contained in the subcopulas.

In the next section, we will summarize the results of these findings. In [Section 3](#), we will focus on the issue of the domain of subcopulas in discrete models. A discussion will also be provided at the end of this work.

2. Concepts and terminologies

In this section, we will provide an overview on concepts and terminologies used throughout this work, focusing on subcopula estimations. First, recall that a *copula* is simply the restriction of a distribution functions with uniform marginals on the unit hypercube $[0, 1]^k$. Since the support of such distribution lies in $[0, 1]^k$, the copula still contain all essential information of that distribution function. Hence, it can also be thought of as a distribution function by abusing the notation. A *subcopula* is simply a restriction of a copula on the domain of the form $\prod_{i=1}^k A_i$ where $0, 1 \in A_i$ for all $i = 1, \dots, k$. Since a copula is continuous,

any subcopula with domain $\prod_{i=1}^k A_i$ can be uniquely extended to a subcopula with domain $\overline{\prod_{i=1}^k A_i}$. Therefore, it is usually required that the domain of a subcopula is closed. It is also possible for a subcopula to be a restriction of distinct copulas. In other words, copula extensions of subcopulas are not unique in general. For characterization of such extensions, see, for example (de Amo *et al.*, 2017).

For convenient, we will define the vector-valued functions \vec{H} and \vec{H}^- associated with a joint distribution function H by letting:

$$\vec{H}(x_1, \dots, x_k) = (H_1(x_1), \dots, H_k(x_k))$$

and

$$\vec{H}^-(x_1, \dots, x_k) = (H_1^-(x_1), \dots, H_k^-(x_k))$$

for all $x_1, \dots, x_k \in \mathbb{R}$ where H_1, \dots, H_k are marginal distribution functions of H and H_i^- is the quantile function of H_i . With this notation, equation (1), for example, can be written succinctly as $H = C \circ \vec{H}$ where the right side denote the function composition of C and \vec{H} .

Sklar's theorem states that any joint distribution function can be written in the form $H = S \circ \vec{H}$ and vice versa where S is a subcopula with domain $\overline{\text{Range}(\vec{H})} = \prod_{i=1}^k \overline{\text{Range}(H_i)}$. In fact, $S = H \circ \vec{H}^-$ on $\text{Dom}(S)$. In this way, the tuple (S, \vec{H}) can be thought of as a representation of the distribution function H that is faithful in the sense that different pairs of (S, \vec{H}) will provide different joint distribution functions.

Notice that \vec{H} only contains the behavior of each random variables. Thus, all information regarding association among random variables must contain in the subcopula S . Thus, S is usually considered to be the dependence structure of the joint distribution function H .

Similar to the true joint distribution function, the true subcopula is unknown and has to be estimated from data. In order to do that, we need a notion of convergence in the space of subcopulas, or equivalently, the notion of distance between two subcopulas. Since the domain of subcopulas varies, we need to consider a representation of subcopulas in a way that their distance can be computed. In other words, we need to embed the set of subcopulas into a metric space. Several works have been done in this direction.

First, de Amo *et al.* (2017) identified a subcopula S with its graph

$$G(S) = \left\{ (\vec{x}, S(\vec{x})) \mid \vec{x} \in \text{Dom}(S) \right\}.$$

They then define the distance ξ via

$$\xi(S, T) = h_{d_\infty}(G(S), G(T))$$

for any subcopulas S and T where d_∞ is the Chebyshev distance and h_d denotes the Hausdorff distance between closed subsets in a metric space with d as its distance function.

Hausdorff distance has also been used by Rachasingho and Tasena (2020) to define a distance between bivariate subcopulas. The idea has been extended to multivariate cases in Tasena (2021b). Denote the class of copulas extending a subcopula S by $[S]$. Rachasingho and Tasena define:

$$\eta(S, T) = h_{d_\infty}([S], [T]) + h_{d_\infty}(\text{Dom}(S), \text{Dom}(T))$$

for all subcopulas S and T . It is proved that η and ξ induced the same topology, that is, convergence in η is the same as convergence in ξ (Tasena, 2021b, Theorem 3.3, Theorem 3.6).

Rachasingho and Tasena (2018), Tasena (2021a) also consider another representation of subcopulas. Recall that in the continuous case, the random variable $F(X)$ is uniform whenever a random variable X has distribution F . Apply this fact in the multivariate setting to a random vector \vec{X} with a continuous joint distribution function H , we can conclude that the random vector $\vec{U} = \vec{H} \circ \vec{X}$ has uniform marginals. In fact, the joint distribution function of \vec{U} is the copula associated with H . So we could argue that the joint distribution function of $\vec{U} = \vec{H} \circ \vec{X}$ is the dependence structure of H and continue to do so even in the noncontinuous case. In this latter case, the joint distribution function of \vec{U} is actually the distribution form of the subcopula associated with the joint distribution H . Here, the *distribution form* S^D of a subcopula S is defined by:

$$S^D(\vec{x}) = \sup\{S(\vec{s}) \mid \vec{s} \leq \vec{x}, \vec{s} \in \text{Dom}(S)\}$$

for all $\vec{x} \in [0, 1]^k$ (Tasena, 2021a). Notice that $S^D = S$ when S is a copula and S is the restriction of S^D on $\text{Dom}(S)$ in general. Therefore, S^D can be treated as a (faithful) representation of S . Since the probability that $\vec{U} = \vec{H} \circ \vec{X}$ belongs to the domain of the subcopula S is one, the extension part of S^D does not really contain any information in the probabilistic sense. Therefore, S^D does not change the dependence structure of the joint distribution function H . The fact that S^D is a distribution function also implies that well-studied modes of convergence for distribution functions can be applied to distribution forms of subcopulas as well. In fact, the Chebyshev distance for distribution forms of subcopulas has been studied in Rachasingho and Tasena (2018) while Levy distance, which metrises the weak convergence, has been studied in Tasena (2021a). The latter has also been proved to metrically equivalent to ξ (Tasena, 2021a, p. 8). For subcopulas S and T , their Levy distance $l(S, T)$ can be written as:

$$l(S, T) = \inf\left\{\epsilon > 0 \mid T^D(\vec{u} - \epsilon \vec{1}) - \epsilon \leq S^D(\vec{u}) \leq T^D(\vec{u} + \epsilon \vec{1}) + \epsilon, \forall \vec{u} \in \mathbb{R}^k\right\}.$$

We summarize the convergence results found in these works again in the following theorem.

Theorem 2.1. Let S_n be a sequence of subcopulas and S be another subcopula with the same dimension. Then the following statements are equivalent.

- (1) The graph of S_n converges to the graph of S under the Hausdorff distance.
- (2) The domain of S_n converges to the domain of S under the Hausdorff distance and the class of copula extensions of S_n converges to the class of copula extensions of S . The latter is equivalent to the following two conditions:
 - if a sequence of copula C_{n_k} extending S_{n_k} converges to a copula C as $n_k \rightarrow \infty$, then C must be a copula extension of S , and
 - for any copula C extending S , there must be a sequence of copula C_n extending S_n such that C_n converges to C .

(3) S_n^D converges weakly to S^D , that is, either one of the following equivalent conditions hold:

- $S_n^D(\vec{u})$ converges weakly to $S^D(\vec{u})$ for any continuity point \vec{u} of S^D ,
- $\int \psi dS_n^D \rightarrow \int \psi dS^D$ for any continuous function ψ ,
- $\limsup_{n \rightarrow \infty} \int 1_K dS_n^D \leq \int 1_K dS^D$ for any closed set K , and
- $\liminf_{n \rightarrow \infty} \int 1_G dS_n^D \geq \int 1_G dS^D$ for any open set G .

Henceforth, we will denote $S_n \rightarrow S$ if a sequence of subcopulas S_n converges to a subcopula S in the sense of the above theorem.

3. Empirical subcopulas in discrete model

In the previous section, we focus on the convergence of subcopulas that lay a groundwork for subcopula estimations. In this section, we will discuss empirical subcopulas in discrete model and show that it is possible to construct an estimator with a specific form of domains according to the marginal distributions.

First, recall the definition of empirical distribution functions. Let $\vec{X}_1, \dots, \vec{X}_n$ be an i.i.d. sample from a k -dimensional distribution function H . Then the empirical distribution H_n associated to this sample is defined by:

$$H_n(\vec{x}) = \frac{1}{n} \sum_{i=1}^n 1_{\{\vec{X}_i \leq \vec{x}\}}$$

for all $\vec{x} \in \mathbb{R}^k$. It is known that H_n is a (random) distribution function with its range being a subset of $\{0, \frac{1}{n}, \dots, 1\}^k$. Thus, H_n is associated with a (random) subcopula S_n with $\text{Dom}(S_n) = \text{Range}(\vec{H}_n) \subseteq \{0, \frac{1}{n}, \dots, 1\}^k$ defined by $S_n = H_n \circ \vec{H}_n^-$. We will call S_n an *empirical subcopula*.

By [Tasena \(2021a, Theorem 4.6\)](#), we know that:

$$l(S_n, S) \leq (2k + 1)d_\infty(H_n, H)$$

where S is the true subcopula associated with H . Thus, we have $S_n \rightarrow S$ since $H_n \rightarrow H$ uniformly.

Next, we will provide some implications of these results in discrete model. Recall that a random variable X is discrete if its support $D = D_X$ is a discrete set, that is, $d_\infty(x, D \setminus \{x\}) > 0$ for all $x \in D$. It follows that such D must be countable and do not have any limit point. For example, if X has binomial distribution, then $D = \{0, 1, \dots, n\}$ while if X is Poisson, then $D = \mathbb{N}$, etc. It follows that the range of its distribution function is also discrete.

Proposition 3.1. Let F be the distribution function of a discrete random variable. Then $\text{Range}(F)$ is a discrete set and $\text{Range}(F) = \text{Range}(F) \cup \{0, 1\}$.

Proof. Let X have distribution F and D be its support. Let $u \in \text{Range}(F)$, that is, $u = F(x)$ for some $x \in D$. Denote $x_- = \sup D \cap (-\infty, x)$ and $x_+ = \inf D \cap (x, \infty)$. Since D is discrete and closed, $x_-, x_+ \in D$ and $x_- < x < x_+$. Denote $u_- = F(x_-)$ and $u_+ = F(x_+)$, then we also

have $u_- = \mathbb{P}(X < x)$ and $u = \mathbb{P}(X < x_+)$. Therefore, $(u_-, u_+) \cap \text{Range}(F) = \{u\}$. Since $x \in D$ we must have $u_- \neq u \neq u_+$ that implies $d_\infty(u, \text{Range}(F) \setminus \{u\}) > 0$ as desired.

Next, suppose that $u_n \rightarrow u \in (0, 1) \setminus \text{Range}(F)$. Since $\text{Range}(F)$ is discrete, we must have $u_n \in (0, 1) \setminus \text{Range}(F)$ for all large n . Thus, u cannot be a limit point of $\text{Range}(F)$. Therefore, $\overline{\text{Range}(F)} \subseteq \text{Range}(F) \cup \{0, 1\}$. Since $0 = \lim_{x \rightarrow -\infty} F(x)$ and $1 = \lim_{x \rightarrow \infty} F(x)$, we also have $0, 1 \in \overline{\text{Range}(F)}$. Thus, $\overline{\text{Range}(F)} = \text{Range}(F) \cup \{0, 1\}$. \square

As a consequence of the above theorem, the limit points of $\text{Dom}(S)$ only lie on the boundary of $[0, 1]^k$. Therefore, $d_\infty(\vec{u}, \text{Dom}(S) \setminus \{\vec{u}\}) > 0$ for all $\vec{u} \in \text{Dom}(S) \cap (0, 1)^k$. Notice that $\text{Dom}(S)$ is finite if its associated random variables all have finite support.

Now, Theorem 2.1 and the fact that $S_n \rightarrow S$ implies $\text{Dom}(S_n) \rightarrow \text{Dom}(S)$ under the Hausdorff distance. Nevertheless, $\text{Dom}(S_n)$ might be undesirable in semi-parametric model where the form of marginal distribution functions are known. In such a case, it is also possible to construct another estimator of subcopulas with the desired domain. We will provide a demonstration for this point in the case of Binomial distributions. Similar arguments can be applied to other discrete distributions.

Denote $R_{m,p}$ the range of the binomial distribution function with parameter $m \in \mathbb{N}$ and $p \in (0, 1)$. Then:

$$R_{m,p} = \{0\} \cup \left\{ \sum_{i=0}^k \binom{m}{i} p^i (1-p)^{m-i} \mid k = 0, \dots, m \right\}.$$

if we define:

$$f_k(p) = \sum_{i=0}^k \binom{m}{i} p^i (1-p)^{m-i}.$$

then

$$\begin{aligned} f'_k(p) &= \sum_{i=0}^k \binom{m}{i} \left(i p^{i-1} (1-p)^{m-i} - (m-i) p^i (1-p)^{m-i-1} \right) \\ &= \sum_{i=0}^k \binom{m}{i} \left(i(1-p) - (m-i)p \right) p^{i-1} (1-p)^{m-i-1} \\ &= \sum_{i=0}^k \binom{m}{i} (i - mp) p^{i-1} (1-p)^{m-i-1} \end{aligned}$$

that implies:

$$f'_k(p) \leq m + m \sum_{i=1}^k \binom{m}{i} \leq m(2^m + 1)$$

when $k < m$. It follows that

$$f_k(p) - f_k(p') \leq m(2^m + 1)p - p'$$

for all $k < m$. Therefore,

$$h_{d_\infty}(R_{m,p}, R_{m,p'}) \leq m(2^m + 1)p - p'$$

also. In particular, $R_{m,p_n} \rightarrow R_{m,p}$ under Hausdorff distance whenever $p_n \rightarrow p$. Note that the number $m(2^m + 1)$ might seem rather large but this is only a rough upper bound for f_k' that is sufficient for the current argument.

Now, assume that the i th marginal distribution of our sample is binomial with parameter $m_i \in \mathbb{N}$ and $p_i \in (0, 1)$ so that $\text{Dom}(S) = R_{\vec{m}, \vec{p}} = \prod_{i=1}^k R_{m_i, p_i}$. In this case, we might want to estimate S with subcopula \hat{S}_n with domain of the same form, say, $\text{Dom}(\hat{S}_n) = R_{\vec{m}, \vec{p}_n}$ where $\vec{p}_n \rightarrow \vec{p}$ a.s. One construction method is as follows.

- (1) Estimate $\vec{p} = (p_1, \dots, p_k)$ with $\vec{p}_n = (p_{1n}, \dots, p_{kn})$ where $p_{jn} = \frac{1}{nm_j} \sum_{i=1}^n X_{ji}$.
- (2) Extend the empirical subcopulas S_n to random copulas C_n . (For characterization of copula extensions, see (de Amo *et al.*, 2017)).
- (3) Set \hat{S}_n to be the restriction of C_n on $R_{\vec{m}, \vec{p}_n}$.

Theorem 3.2. The (random) subcopula \hat{S}_n constructed above is (weakly) consistent, that is, $\hat{S}_n \rightarrow S$ a.s.

Proof. It is well-known that $\vec{p}_n \rightarrow \vec{p}$ a.s. Thus, we have $\text{Dom}(\hat{S}_n) \rightarrow \text{Dom}(S)$ under the Hausdorff distance. Since \hat{S}_n and S_n share a common copula extension,

$$l(\hat{S}_n, S_n) \leq kh_{d_\infty}(\text{Dom}(\hat{S}_n), \text{Dom}(S_n))$$

by Tasena (2021a, Proposition A.4). Note that the right side converges to zero since $\text{Dom}(S_n) \rightarrow \text{Dom}(S)$ also under the Hausdorff distance. Now, the fact that $S_n \rightarrow S$ implies $\hat{S}_n \rightarrow S$ as desired.

4. Conclusion and discussion

In this work, we discuss subcopula estimation in discrete models. We summarize the results discovered recently regarding convergences of subcopulas focusing on weak convergences of distribution functions. It is known that empirical subcopulas weakly converge to the true subcopula. We also construct another subcopula estimator in the case where the marginal distributions of random vectors are known. For example, in the case that each random variable has Binomial distribution. While empirical subcopulas, in this case, might not correspond to those that have Binomial distributions as their marginals. This new subcopula estimator does possess such property. We also argue that it is better to use subcopulas instead of copulas in a discrete model. There are a few works sharing our opinion, see for example (Faugeras, 2017; Geenens, 2020; Trivedi and Zimmer, 2017). See also Nikoloulopoulos (2013) for the problem that might arise when using copula to model discrete data.

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