

On the statics of curved masonry structures via numerical models

Curved
masonry
structures

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Abstract

Purpose – This paper aims to review recent literature results on the equilibrium problem and the strengthening design of masonry vaults.

Design/methodology/approach – A Lumped Stress Method (LSM) is considered within the Heyman's safe theorem, based on the definition of thrust surface of a masonry curved structure. In particular, the static problem of the vault is formulated by introducing a membrane continuous of the studied masonry structure to associate with a spatial truss through a nonconforming variational approximation of the thrust surface and membrane stress potential. A tensegrity approach based on a minimal mass design strategy, different strengths in tension and compression of the material is discussed within the strengthening strategy of masonry vaults.

Findings – The numerical results have highlighted the efficacy of the two numerical approaches to assess the vulnerability of existing structures and design optimal strengthening interventions of these structures.

Originality/value – The presented models can represent fast and useful tools to assess the vulnerability of existing structures and design optimal strengthening interventions with composite materials of these structures.

Keywords Lumped stress method, Masonry structures, Structural strengthening, Tensegrity approach, Vaults

Paper type Review paper

1. Introduction

The study of structural vulnerability of masonry historical constructions is an international priority to evaluate their safe and to preserve them over time by designing rehabilitation interventions.

Several mechanical models available in literature are based on either local failure mechanisms (Fraternali, 2007; Berardi, 2016) or the Heyman limit analysis approach (Heyman, 1995). With reference to arches, the safe theorem of Heyman states that the structure is safe if a line of thrust (funicular curve) in equilibrium with the external loads and entirely contained within the volume of the structure can be found.

This theorem has been extended to double-curved structures via either continuous (Baratta and Corbi, 2011; Baratta and Corbi, 2013) or discontinuous (Fraternali *et al.*, 2002; Fraternali, 2010, 2011; Carpentieri *et al.*, 2016) approaches. The theoretical approach and the

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computational burden of such models makes their implementation in finite element method (FEM) code inappropriate.

This paper reviews an advanced numerical model capable to search a “safe” thrust surface of masonry vaults, via an iterative procedure based on a constrained lumped stress method (LSM) (Fraternali *et al.*, 2002; Fraternali, 2010, 2011) and a tensegrity approach capable to optimize the design of strengthening systems on masonry vaults (Fraternali *et al.*, 2015; Fraternali, 2007).

The first model considers that such structures exhibit a no-tension membrane state of stress across the thrust surface. The membrane equilibrium is used to research an optimal shape of the surface of the masonry model, under specific no-tension and position constraints. The Pucher’s formulation is applied to identify the potential of the membrane stress, by a nonconforming variational approach, with defined polyhedral approximations. It leads to a representation of such a stress field by a discrete network of compressive forces and allows to predict the regions exposed to crack pattern and to evaluate a lower bound of the collapse load of examined existing vault. This model allows to evaluate the statics of existing masonry structures and their vulnerability under external loads.

The tensegrity approach allows to design optimal strengthening interventions with fiber reinforced polymers (FRP) or fabric reinforced cementitious matrix (FRCM) to upgrade and retrofit existing structures. This approach starts by modeling a structure as tensegrity networks of masonry struts and tensile elements corresponding to the strengthened regions of masonry. These reinforcements are generally represented by unidirectional composites, applied to masonry structures along fixed directions (Mazzotti *et al.*, 2015; Carozzi and Poggi, 2015; Baratta and Corbi, 2015; Angelillo *et al.*, 2014; Fabbrocino *et al.*, 2015; De Piano *et al.*, 2017). The proposed optimization strategy searches the minimal mass tensegrity structure connecting a given node set, by assuming different yielding constraints on compressive (masonry) and tensile (FRP/FRCM) elements.

An initial connection pattern is defined by potential connections of each node to all the neighbor nodes lying in a spheroidal domain of fixed radius through compressive and tensile elements. The minimal mass optimization procedure allows to obtain a minimal mass resisting mechanism of the reinforced structure via the relocation of nodes and the upgrade of connections, accounting different strengths for the masonry struts and the FRP/FRCM reinforcements, which can be regarded as an alternative LS/thrust network model of the examined structure. The stress field given by the procedure is statically admissible, by assuming the assumption of stable plastic response of masonry in compression and reinforcements in tension, and safe theorem of the limit analysis of elastic-plastic bodies (Koiter, 1960) ensures that the reinforced structure is safe under the examined loading conditions.

We continue by reviewing some case studies for the research of the thrust surface of masonry vaults and the optimal design of strengthening interventions.

2. Lumped stress method

2.1 Mechanical model

LS models of vaulted structures have been formulated in recent studies through polyhedral stress functions, i.e. piecewise linear functions $\hat{\varphi}$ defined over triangulations Π_h of a simply-connected domain Ω , which lies in the x-y plane of a given Cartesian frame (Fraternali *et al.*, 2002; Fraternali, 2010, 2011). These models are based on the assumption that an internally self-equilibrated systems of forces applied on to the edges of Π_h can be associated to scalar functions defined over the same mesh, with the property that the generic force \hat{P}_{ij}

corresponds to the jump in the normal derivative of $\hat{\varphi}$ across the edge connecting nodes i and j . In particular, concave “folds” of $\hat{\varphi}$ generate compressive forces, while “convex” folds generate tensile forces (refer to the illustrative example shown in [Figure 1](#)).

It is worth noting that the existence of a solution to the equilibrium problem of the vault guarantees its stability under the given loading, in the light of the static theorem of limit analysis for masonry structures.

2.2 Iterative form-finding procedure

The proposed LSM is described as follows:

- *Step 1:* It is assumed that an initial geometry $\hat{\mathbf{f}} = \hat{\mathbf{f}}_1$ of the thrust surface (the middle surface of the curved structure).
- *Step 2:* For $\hat{\mathbf{f}} = \hat{\mathbf{f}}_1$, find the stress function vector $\hat{\boldsymbol{\varphi}}_1$ that solves the equilibrium equations and the boundary conditions $\hat{\boldsymbol{\varphi}}_1 = \hat{\boldsymbol{\varphi}}''$ on S''_h .
- *Step 3:* Compute the convex hull of $\hat{\boldsymbol{\varphi}}_1$ and consider the concave (upper) edge of such a region (“concave hull” of $\hat{\boldsymbol{\varphi}}_1$), obtaining a no-tension stress function $\hat{\boldsymbol{\varphi}}_2$.
- *Step 4:* Project $\hat{\boldsymbol{\varphi}}_2$ onto the original triangulation Π_h , through linear interpolation, obtaining a new stress function $\hat{\boldsymbol{\varphi}}_3$; for $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}_3$, compute the geometry vector $\hat{\mathbf{f}}_3$ that solves the equilibrium equations (3) under the boundary conditions $\hat{\mathbf{f}}_3 = \hat{\mathbf{f}}''$ on S''_h .
- *Step 5:* Set $\hat{\mathbf{f}}_1 = \hat{\mathbf{f}}_3$ and return to Step 2.

The final goal of the above solution strategy consists of a suitable topology optimization of the adopted truss model.

3. Tensegrity model

3.1 Mechanical model

Let us consider a masonry vault or dome with mean surface described by a set of n_n nodes in the 3D Euclidean space ([Fraternali et al., 2015](#)), whose position vectors \mathbf{n}_k are referred to a given cartesian frame $\{O, x, y, z\}$. Let us introduce the node matrix, \mathbf{N} , given by the components (x_k, y_k, z_k) of the position vectors of all nodes as follows:

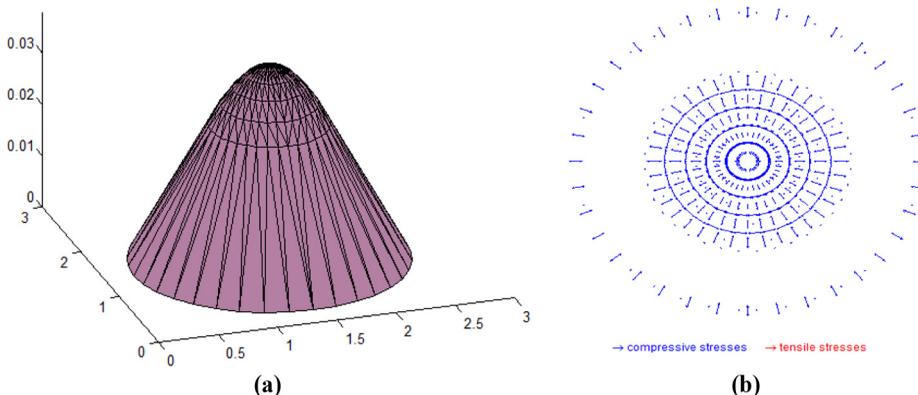


Figure 1.
Illustration of a
polyhedral stress
function (a) and the
associated system of
planar forces (b)

$$\mathbf{N} = \begin{bmatrix} x_1 & \dots & \dots & \dots & x_{n_n} \\ x_2 & \dots & \dots & \dots & y_{n_n} \\ x_3 & \dots & \dots & \dots & z_{n_n} \end{bmatrix} \quad (1)$$

The initial connection pattern (Figure 2) is modeled by fixing the connection radius, r_k and connecting the node k -th with all the j -th neighbors nodes under the following condition:

$$|\mathbf{n}_k - \mathbf{n}_j| \leq r_k \quad (2)$$

The connection of the k -th node to the j -th neighbor node is realized through two elements working in parallel: a compressive masonry strut (or bar) $\mathbf{b}_i = \mathbf{n}_k - \mathbf{n}_j$ and a tensile FRP/FRCM element (or string) $\mathbf{s}_i = \mathbf{n}_k - \mathbf{n}_j$.

Let us assume λ_i and γ_j , the compressive force per unit length (force density) acting in the i -th bar and the tensile force per unit length acting in the j -th string, both defined to be positive quantities; n_b and n_s the total number of bars and of strings composing the background structure, respectively (with $n_b = n_s$ in the initial configuration); $\mathbf{x} = [\lambda_1 \dots$

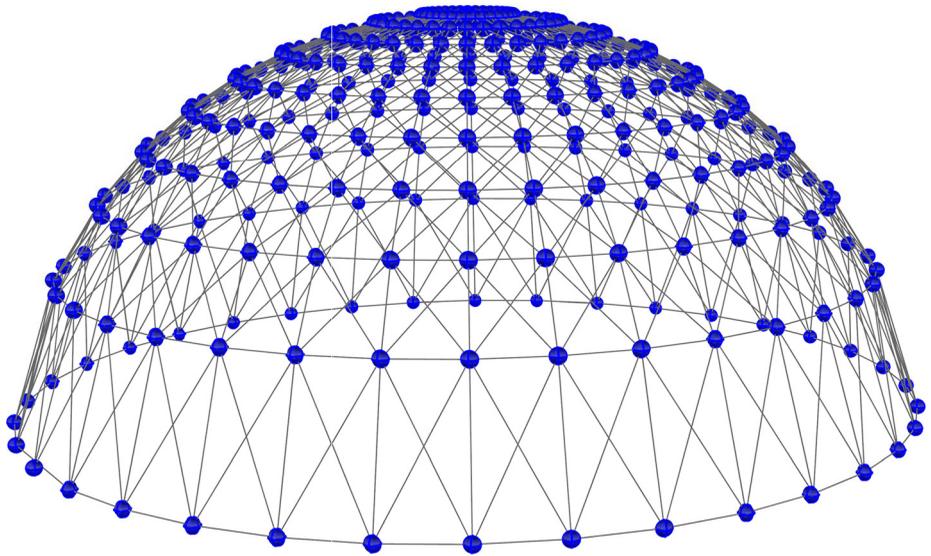


Figure 2.
Initial connection
pattern

$\lambda_{nb} \gamma_1 \dots \gamma_{ns}]^T$ the vector with $n_x = (n_b + n_s)$ entries that collect the force densities in bars and strings, \mathbf{w} the external load vector.

The static equilibrium equations under a given load condition is expressed as follows:

$$\mathbf{Ax} = \mathbf{w} \quad (3)$$

where \mathbf{A} is the static matrix of the structure ($3n_n \times n_x$), referred to the geometry and the connectivity of bars and strings (Nagase and Skelton, 2014).

The materials constraints are imposed by assuming elastic-perfectly-plastic bars and strings via the following inequalities:

$$\lambda_i b_i \leq \sigma_{bi} A_{bi} \quad (4)$$

$$\gamma_j s_j \leq \sigma_{si} A_{si} \quad (5)$$

being σ_{bi} and σ_{si} the compressive strength of the generic bar and the tensile strength of the generic string, respectively.

The masses of each bar and string depend on the mass densities of bar, ρ_{bi} , and string, ρ_{si} and are given by the following expressions:

$$m_{bi} = \rho_{bi} A_{bi} b_i \quad (6)$$

$$m_{si} = \rho_{si} A_{si} s_i \quad (7)$$

3.2 Minimization approach

The minimal mass design of the background structure is performed by the following linear program (Fraternali *et al.*, 2015; Fraternali, 2007):

$$\begin{aligned} & \underset{\mathbf{x}, \mathbf{y}}{\text{minimize}} && m = \mathbf{d}^T \mathbf{y} \\ & \text{subject to} && \begin{cases} \mathbf{Ax} = \mathbf{w} \\ \mathbf{Cx} \leq \mathbf{Dy} \\ \mathbf{x} \geq 0, \mathbf{y} \geq 0 \end{cases} \end{aligned} \quad (8)$$

being

$$\mathbf{y} = [A_{b1} \dots A_{bnb} A_{s1} \dots A_{sns}]^T$$

$$\mathbf{d}^T = [r_{bi} b_i \dots r_{bnb} b_{nb} r_{si} s_i \dots r_{snb} s_{ns}]^T$$

$$\mathbf{C} = \begin{bmatrix} \text{diag}(b_1, \dots, b_{nb}) & \mathbf{0} \\ \mathbf{0} & \text{diag}(s_1, \dots, s_{ns}) \end{bmatrix}$$

$$D = \begin{bmatrix} \text{diag}(\sigma_{b1}, \dots, \sigma_{bnb}) & \mathbf{0} \\ \mathbf{0} & \text{diag}(\sigma_{s1}, \dots, \sigma_{sns}) \end{bmatrix}$$

The solution to problem in [equation \(8\)](#) provides minimal-mass configuration of the background structure, chooses whether a bar or a string connects each couple of interacting nodes and returns bars and strings with zero cross-section areas in correspondence with the interacting nodes that do not need to be connected in the minimal mass configuration, under the given equilibrium and yielding constraints.

4. Case studies

4.1 Numerical results of lumped stress method

The example deals with a cloister vault of [Figure 3](#) subject to a force applied $p = 20$ kN.

The base of the cloister vault is restrained by fixed hinge supports. The examined vault is characterized as follows: a constant thickness of 0.11 m; a side of the vault of 2 m; a height of 1 m; and a self-weight (gm) of 20 kN/m³.

The final solution highlights a thrust surface not contained between intrados and extrados of the vault and, then, the structure is not safe ([Figure 4](#)).

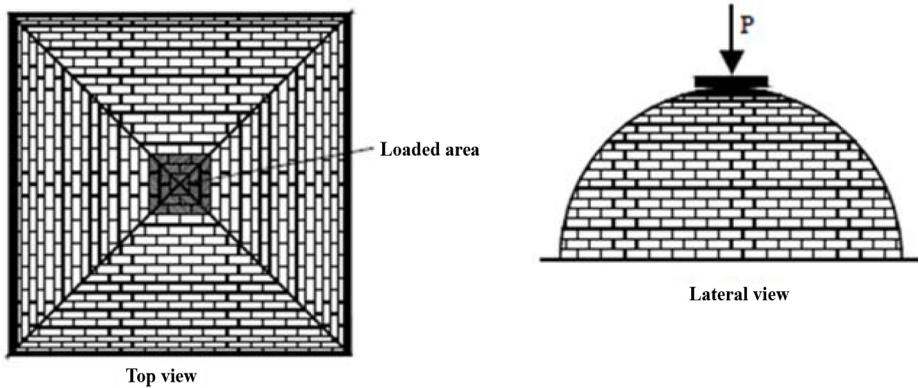
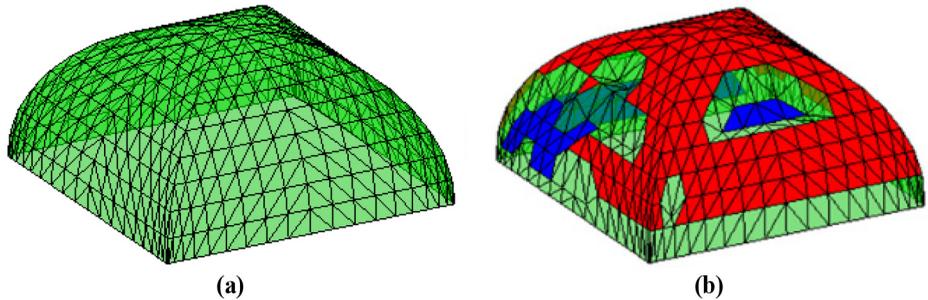


Figure 3.
Load condition of the cloister vault

Figure 4.

(a) Initial 3D-triangulation of the thrust surface; (b) final solution of the trust surface—red regions = area external to extrados; blue regions = area external to intrados



4.2 Numerical results of tensegrity model

The case study presented in Fraternali *et al.* (2015) deals with a tufo brick masonry with 15.0 kN/m^3 self-weight and 13 MPa compressive strength, externally reinforced with FRP and FRCM strips.

The composite tensile strength is assumed equal to 376.13 MPa and the composite thickness equal to 0.17 mm.

The geometry of the examined vault is illustrated in Figure 5 of Fraternali *et al.* (2015), together with the corresponding background structure, which features 441 nodes and 4,508 connections.

The optimal reinforcement of such a vault under vertical loading is mainly formed by parallel FRP/FRCM strips 82 mm maximum width near the crown. The above reinforcements are integrated with diagonal FRP/FRCM strips with about 140 mm maximum width near the intersections of the four vault segments, under combined vertical and seismic loading [Figure 5 of Fraternali *et al.* (2015)].

The analyzed seismic loading consists of horizontal forces with magnitude equal to 0.35 of the magnitude of vertical forces in all nodes, which approximate the effects of a seismic excitation of the examined structure through a conventional static approach. The compressed network includes couples of diagonal arches near the corners, parallel-line arches and diagonal struts over the vault segments [Figure 5 of Fraternali *et al.* (2015)].

5. Conclusions

We have reviewed recent literature studies on the equilibrium problem and the strengthening design of masonry vaults. More specifically, the LSM allows to model the membrane state of stress carried by masonry structures through an adaptive, predictor-corrector technique. The approach provides statically admissible force network according with no-tension constraints and allows to evaluate if a vault is safe or not under fixed load conditions.

The tensegrity approach allows to analyze masonry structures of general shape and dimensions, including structural complexes formed by an arbitrary combination of walls, vaults and domes. The adopted optimization approach gives noninvasive reinforcement patterns, which can be able to preserve a sufficient crack-adaption capacity of the structure, under the respect of the equilibrium equations and material yield limits.

The numerical results have highlighted the efficacy of the two numerical approaches that could represent fast and useful tools to assess the vulnerability of existing structures and design optimal strengthening interventions with composite materials of these structures. Future developments of the present work will be focused on exploring alternative reinforcement systems (Barretta *et al.*, 2017).

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Further reading

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