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Foundational aspects of a new matrix holomorphic structure

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Abstract

Purpose – In this paper we talk about complex matrix quaternions (biquaternions) and we deal with some abstract methods in mathematical complex matrix analysis.

Design/methodology/approach – We introduce and investigate the complex space $\mathbb{H}_{\mathbb{C}}$ consisting of all 2×2 complex matrices of the form $\xi = \begin{pmatrix} z_1 + iw_1 & z_2 + iw_2 \\ -\bar{z}_2 - i\overline{w}_2 & \bar{z}_1 + i\overline{w}_1 \end{pmatrix}$, $(z_1, w_1, z_2, w_2) \in \mathbb{C}^4$.

Findings – We develop on $\mathbb{H}_{\mathbb{C}}$ a new matrix holomorphic structure for which we provide the fundamental operational calculus properties.

Originality/value – We give sufficient and necessary conditions in terms of Cauchy–Riemann type quaternionic differential equations for holomorphicity of a function of one complex matrix variable $\xi \in \mathbb{H}_{\mathbb{C}}$. In particular, we show that we have a lot of holomorphic functions of one matrix quaternion variable.

Keywords Complex quaternion, Complex quaternionic space, Quaternionic holomorphic function, Cauchy–Riemann equations

Paper type Research paper

1. Introduction

The theory of quaternionic analysis was founded in 1928 and is devoted especially to the study of the so-called regular functions introduced by R. Fueter in 1935 [1] which satisfy the (left) Cauchy–Fueter equation

$$\frac{\partial}{\partial x_1}(.) + i\frac{\partial}{\partial x_2}(.) + j\frac{\partial}{\partial x_3}(.) + k\frac{\partial}{\partial x_4}(.) = 0,$$

where $\{1, i, j, k\}$ is the standard basis of the four-dimensional real algebra $\mathbb{H}$ of the quaternions numbers constructed in 1843 by W. R. Hamilton [2]. All such quaternion numbers have the representation

$$x_0 + x_1 i + x_2 j + x_3 k, \quad (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$$

which provide in fact a foundation for simple mathematical representation of rotations. Therefore, they are powerfully used in the fields of mechanics, magnetism, aerospace, software development, etc. Thus, many mathematicians show a great interest in studying

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quaternionic analysis and particularly quaternionic analysis over a complex structure. Among many research papers about quaternionic analysis, for instance, we may observe the various versions of their works presented in Refs. [3–5].

In this paper, we are deeply interested in the field $\mathcal{M}_{\mathbb{H}}$ of quaternion numbers represented by all 2×2 complex matrices having the form

$$z=\begin{pmatrix} z_1 & z_2\\ -\bar{z}_2 & \bar{z}_1\end{pmatrix},\ (z_1,z_2)\in\mathbb{C}^2.$$

such $\mathcal{M}_{\mathbb{H}}$ will be regarded here as an $\mathbb{R}$-algebra isomorphic to the $\mathbb{R}$-algebra $\mathbb{H}$ of W. R. Hamilton. We propose to introduce the complex space $\mathbb{H}_{\mathbb{C}}$ of the so-called complex quaternion numbers of the form

$$\xi=z+iw,\,(z,w)\in\mathcal{M}_{\mathbb{H}}\times\mathcal{M}_{\mathbb{H}},$$

so that

$$\mathbb{H}_{\mathbb{C}}=\left\{\xi=\begin{pmatrix} z_1+iw_1 & z_2+iw_2\\ -\bar{z}_2-i\overline{w}_2 & \bar{z}_1+i\overline{w}_1\end{pmatrix}:\ (z_1,w_1,z_2,w_2)\in\mathbb{C}^4\right\}.$$

We present $\mathbb{H}_{\mathbb{C}}$ as the left $\mathcal{M}_{\mathbb{H}}$-vector space of complex quaternion numbers with basis

$$\{\mathbb{1},i\mathbb{1}\}=\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}i&0\\0&i\end{pmatrix}\right\}.$$

so that $\mathbb{H}_{\mathbb{C}}$ has a splitting into the direct sum $\mathcal{M}_{\mathbb{H}}\oplus i\mathcal{M}_{\mathbb{H}}$. Moreover, we develop on $\mathbb{H}_{\mathbb{C}}$ a new matrix holomorphic structure for which we provide the fundamental operational calculus properties.

The main parts of this work are organized as follows. In Section 2, we present the construction of the space $\mathbb{H}_{\mathbb{C}}$ of the complex quaternion numbers and we prove the following.

Theorem 2.1. Let $(\mathbb{H}_{\mathbb{C}},+,.)$ *denote the set* $\mathcal{M}_{\mathbb{H}}\times\mathcal{M}_{\mathbb{H}}$ *equipped with the operations* $(+)$ *and* $(.)$ *defined such that for all* $(z,w),(z',w')\in\mathbb{H}_{\mathbb{C}}$, for all $\lambda=a+ib\in\mathbb{C}$, *we have:*

(1) $(z,w)+(z',w')=(z+z',w+w')$.

(2) $\lambda.(z,w)=(a+ib)(z,w)=(az-bw,aw+bz)$.

Then, it holds that

(1) $(\mathbb{H}_{\mathbb{C}},+,.)$ *is a* $\mathbb{C}$*-vector space.*

(2) $\mathbb{H}_{\mathbb{C}}$ *has a splitting into a direct sum* $\mathbb{H}_1\oplus i\mathbb{H}_1$ *where* $\mathbb{H}_1$ *is an* $\mathbb{R}$*-sub-vector space of* $\mathbb{H}_{\mathbb{C}}$ *isomorphic to* $\mathcal{M}_{\mathbb{H}}$.

(3) *If* $(\times)$ *denotes the usual multiplication of square matrices, then the space* $(\mathbb{H}_{\mathbb{C}},+,.,\times)$ *is a* $\mathbb{C}$*-algebra and the space* $(\mathcal{M}_{\mathbb{H}},+,.,\times)$ *is an* $\mathbb{R}$*-sub-algebra.*

The above structure on $\mathbb{H}_{\mathbb{C}}$ has its own particular features, it induces a $\mathbb{C}$-algebra structure. So, we have included in this section the basic correspondent algebraic properties. Moreover, we define a conjugation in $\mathbb{H}_{\mathbb{C}}$ for the one $\mathbb{H}_{\mathbb{C}}$ can be viewed as an inner product space. In Section 3, we give the fundamental operational calculus on functions of one complex quaternion (or complex matrix) variable that take values in a vector space $E\in\{\mathbb{R},\mathbb{C},\mathcal{M}_{\mathbb{H}},\mathbb{H}_{\mathbb{C}}\}$. In particular, the concepts of real and complex quaternionic derivatives are introduced. In Section 4, the meaning of a quaternionic holomorphic function is given due to the following operators

$$\partial_\xi =: \partial = \frac{\partial}{\partial z} - i\frac{\partial}{\partial w} \quad \text{and} \quad \overline{\partial}_\xi =: \overline{\partial} = \frac{\partial}{\partial \overline{z}} - i\frac{\partial}{\partial \overline{w}}$$

which act on differentiable quaternionic functions of one variable $\xi \in \mathbb{H}_{\mathbb{C}}$. Therefore, we provide a characterization of quaternionic holomorphic functions by means of sufficient and necessary conditions in terms of Cauchy–Riemann type equations.

Theorem 4.1. Let $E \in \{\mathbb{R}, \mathbb{C}, \mathcal{M}_{\mathbb{H}}, \mathbb{H}_{\mathbb{C}}\}$ and $\Phi : \mathcal{D} \to E$ be a complex quaternionic function of one complex quaternion variable $\xi = z + iw$, with $(z, w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$, defined on an open subset $\mathcal{D}$ in $\mathbb{H}_{\mathbb{C}}$. Suppose that

$$\Phi(\xi) = \Phi(z + iw) = f\left(z, \overline{z}, w, \overline{w}\right) + ig\left(z, \overline{z}, w, \overline{w}\right).$$

Then, Φ is holomorphic on $\mathcal{D}$, if and only if the following Cauchy–Riemann type equations $\frac{\partial f}{\partial \overline{z}} = \frac{\partial g}{\partial \overline{w}}$ and $\frac{\partial f}{\partial \overline{w}} = -\frac{\partial g}{\partial \overline{z}}$ are satisfied.

Such new version of complex structure gives an other way of studying quaternionic analysis. In addition, it is quite different from which provided in Ref. [4] and can be viewed from a complex matrix analysis viewpoint. Furthermore, several different concrete computational methods provided throughout this work show that the presented matrix (quaternionic) complex structure is flexible and is close to the standard complex structure. In fact, this can be shown with the help of Theorem 4.1, providing a non trivial example of holomorphic quaternionic function.

Theorem 4.3. Let Φ: $\xi \mapsto \xi^{-1}$ be the complex quaternionic inversion function defined for all $\xi \in \mathbb{H}_{\mathbb{C}} \setminus S$, where $S = \{\xi \in \mathbb{H}_{\mathbb{C}} : \det\xi = 0\}$. Then, it holds that,

(1) *Φ has a decomposition into $f + ig$ where f and g are functions of two variables $(z, w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$ satisfying the Cauchy–Riemann type equations $\frac{\partial f}{\partial \overline{z}} = \frac{\partial g}{\partial \overline{w}}$ and $\frac{\partial f}{\partial \overline{w}} = -\frac{\partial g}{\partial \overline{z}}$.*

(2) *Φ is a biholomorphism from $\mathbb{H}_{\mathbb{C}} \setminus S$ to $\mathbb{H}_{\mathbb{C}} \setminus S$.*

Theorem 4.3 shows that we have a lot of holomorphic functions of one matrix variable. In fact, the complex matrix analysis is the theory of such functions. The other results of this paper can offer potential methods and stimulate activity in the theory of complex quaternionic analysis. On the other hand, we illustrate our abstract study by several examples to insist that the presented quaternionic holomorphic structure can induce a new aspect of pluripotential theory in quaternionic plurisubharmonic functions as provided in Ref. [6]. Moreover, it should be denoted that our paper can be useful for authors working on subjects studied in Refs. [7–9].

Finally, let us recall that according to Ref. [2], the algebra of quaternion numbers is the non-commutative field

$$\mathbb{H} = \mathbb{R}.1 \oplus \mathbb{R}.i \oplus \mathbb{R}.j \oplus \mathbb{R}.k,$$

which has a structure of a four-dimensional $\mathbb{R}$-vector space, with basis $\{1, i, j, k\}$, for which a binary composition law is equipped and defined as a bilinear form, such that 1 is the unity and

$$i^2 = j^2 = k^2 = -1, \; ij = -ji = k, \; jk = -kj = i, \; ki = -ik = j.$$

on the other hand, the field $\mathbb{H}$ can be described as the sub-algebra $\mathcal{M}_{\mathbb{H}}$ of the $\mathbb{R}$-algebra $\mathcal{M}_2(\mathbb{C})$ of dimension 8 consisting of all complex square matrices (see Ref. [10]). In addition, the isomorphism

$$\begin{array}{ccc} \mathbb{H} & \to & \mathcal{M}_{\mathbb{H}} \\ a + bi + cj + dk & \mapsto & a\mathbb{1} + bJ + cK + dL \end{array}$$

between $\mathbb{H}$ and $\mathcal{M}_{\mathbb{H}}$, shows that $\mathcal{M}_{\mathbb{H}}$ is a 4-dimensional $\mathbb{R}$-vector space with basis $\{\mathbb{1},J,K,L\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}$.

2. The complex quaternionic space $\mathbb{H}_{\mathbb{C}}$

With a complexification method applied on the product $\mathbb{R}$-vector space $\mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$, we shall introduce the complex quaternionic space $\mathbb{H}_{\mathbb{C}}$to be the $\mathbb{C}-$vector space consisting of all 2×2 complex matrices written uniquely as $\xi = z + iw, \quad (z,w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$.

Theorem 2.1. Let $(\mathbb{H}_{\mathbb{C}}, +, .)$ *denote the set* $\mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$ *equipped with the operations* $(+)$ *and* $(.)$ *defined such that for all* $(z,w),(z',w') \in \mathbb{H}_{\mathbb{C}}$, *for all* $\lambda = a + ib \in \mathbb{C}$, *we have:*

- $(z,w) + (z',w') = (z+z', w+w')$.
- $\lambda.(z,w) = (a+ib)(z,w) = (az - bw, aw + bz)$.

Then, it holds that

(i) $(\mathbb{H}_{\mathbb{C}}, +, .)$ *is a* $\mathbb{C}$*-vector space.*

(ii) $\mathbb{H}_{\mathbb{C}}$ *has a splitting into a direct sum* $\mathbb{H}_1 \oplus i\mathbb{H}_1$ *where* $\mathbb{H}_1$ *is an* $\mathbb{R}$*-sub-vector space of* $\mathbb{H}_{\mathbb{C}}$ *isomorphic to* $\mathcal{M}_{\mathbb{H}}$.

(iii) *If* $(\times)$ *denotes the usual multiplication of square matrices, then the space* $(\mathbb{H}_{\mathbb{C}}, +, ., \times)$ *is a* $\mathbb{C}$*-algebra and the space* $(\mathcal{M}_{\mathbb{H}}, +, ., \times)$ *is an* $\mathbb{R}$*-sub-algebra of* $(\mathbb{H}_{\mathbb{C}}, +, ., \times)$.

Proof. Statement (i) holds since the followings are immediately satisfied.

(1) $(\mathbb{H}_{\mathbb{C}}, +)$ is an Abelian group with zero element $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(2) The map $\begin{array}{ccc} \mathbb{C} \times \mathbb{H}_{\mathbb{C}} & \to & \mathbb{H}_{\mathbb{C}} \\ (\lambda, \xi) & \mapsto & \lambda.\xi \end{array}$ satisfies the following rules

(a) $\forall\ (\alpha,\beta) \in \mathbb{C}^2, \forall\ \xi \in \mathbb{H}_{\mathbb{C}}: \quad \alpha.(\beta.\xi) = (\alpha\beta).\xi$.

(b) $\forall\ \xi \in \mathbb{H}_{\mathbb{C}}:\ 1.\xi = \xi$.

(3) $\forall\ \lambda \in \mathbb{C},\ \forall\ (\xi_1,\xi_2) \in \mathbb{H}_{\mathbb{C}} \times \mathbb{H}_{\mathbb{C}}: \quad \lambda.(\xi_1 + \xi_2) = \lambda.\xi_1 + \lambda.\xi_2$.

(4) $\forall\ (\lambda,\mu) \in \mathbb{C}^2,\ \forall\ \xi \in \mathbb{H}_{\mathbb{C}}: \quad (\lambda + \mu).\xi = \lambda.\xi + \mu.\xi$.

Statement (ii) holds since each of the following maps

$$\begin{array}{cccc} \varphi: & \mathcal{M}_{\mathbb{H}} & \to & \mathbb{H}_{\mathbb{C}} \\ & z & \mapsto & z + i0 \end{array} \quad \text{and} \quad \begin{array}{cccc} i\varphi: & \mathcal{M}_{\mathbb{H}} & \to & \mathbb{H}_{\mathbb{C}} \\ & w & \mapsto & 0 + iw \end{array}$$

is $\mathbb{R}$-linear and injective. We let $\mathbb{H}_1 = \varphi(\mathcal{M}_{\mathbb{H}})$ and $i\mathbb{H}_1 = i\varphi(\mathcal{M}_{\mathbb{H}})$. Then, we verify at once that $\mathbb{H}_1 \cap i\mathbb{H}_1 = \{0\}$ and $\mathbb{H}_1 + i\mathbb{H}_1 = \mathbb{H}_{\mathbb{C}}$. The first part of the statement (iii) holds since by statement (i), $(\mathbb{H}_{\mathbb{C}}, +, .)$ is a $\mathbb{C}$-vector space and the multiplication law $(\times)$ in $\mathbb{H}_{\mathbb{C}}$ is associative with unity $\mathbb{1}$. Furthermore, the multiplication law $(\times)$ in $\mathbb{H}_{\mathbb{C}}$ is distributive over the addition law $(+)$. Moreover, for all complex quaternion numbers $(\xi,\eta) \in \mathbb{H}_{\mathbb{C}} \times \mathbb{H}_{\mathbb{C}}$ and for any complex number $\lambda \in \mathbb{C}$, we have $(\lambda.\xi) \times \eta = \xi \times (\lambda.\eta) = \lambda.(\xi \times \eta)$. The second part of the statement (iii) holds since for all $(z,w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$ and for any real number $\lambda \in \mathbb{R}$, we have $z + w \in \mathcal{M}_{\mathbb{H}}, \quad z \times w \in \mathcal{M}_{\mathbb{H}}$ and $\lambda.z \in \mathcal{M}_{\mathbb{H}}$. In addition $\mathbb{1} \in \mathcal{M}_{\mathbb{H}}$, where $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the the square unity matrix. $\square$

2.1 Basic algebraic properties of the space $\mathbb{H}_{\mathbb{C}}$

Proposition 2.2. *Each of the following statements holds in $\mathbb{H}_{\mathbb{C}}$.*

(i) *For all $\xi \in \mathbb{H}_{\mathbb{C}}$, there exists a unique $(z, w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$ such that $\xi = z + iw$. As such $\mathbb{H}_{\mathbb{C}}$ can be identified to the direct sum $\mathcal{M}_{\mathbb{H}} \oplus i\mathcal{M}_{\mathbb{H}}$.*

(ii) *The space $\mathbb{H}_{\mathbb{C}}$ is formed by all 2×2 complex matrices of the form $\xi = \begin{pmatrix} z_1 + iw_1 & z_2 + iw_2 \\ -\overline{z}_2 - i\overline{w}_2 & \overline{z} + i\overline{w}_2 \end{pmatrix}$ where $(z_1, w_1, z_2, w_2) \in \mathbb{C}^4$.*

(iii) *The space $\mathbb{H}_{\mathbb{C}}$ is 4-dimensional and admits the family*

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\} \text{ as a basis over } \mathbb{C}.$$

Proof. Statement (i) holds since the map $\Phi : \begin{array}{ccc} \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}} & \to & \mathbb{H}_{\mathbb{C}} \\ (z, w) & \mapsto & z + iw \end{array}$ is $\mathbb{R}$-linear and injective between finite dimensional $\mathbb{R}$-vector spaces. Indeed, let $z = \begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 & w_2 \\ -\overline{w}_2 & \overline{w}_1 \end{pmatrix}$ be in $\mathcal{M}_{\mathbb{H}}$, $(z_1, w_1, z_2, w_2) \in \mathbb{C}^4$. By a direct computation, we have

$$\begin{aligned} z + iw &= \begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix} + i \begin{pmatrix} w_1 & w_2 \\ -\overline{w}_2 & \overline{w}_1 \end{pmatrix} \\ &= \begin{pmatrix} z_1 + iw_1 & z_2 + iw_2 \\ -\overline{z}_2 - i\overline{w}_2 & \overline{z}_1 + i\overline{w}_1 \end{pmatrix}. \end{aligned}$$

In addition, the equation $z + iw = 0_{\mathbb{H}_{\mathbb{C}}}$ is equivalent to the followings

$$\begin{cases} z_1 + iw_1 = 0 \\ \overline{z}_1 + i\overline{w}_1 = 0 \\ z_2 + iw_2 = 0 \\ \overline{z}_2 + i\overline{w}_2 = 0 \end{cases} \Leftrightarrow \begin{cases} z_1 = -iw_1 \\ z_1 = iw_1 \\ z_2 = -iw_2 \\ z_2 = iw_2 \end{cases} \Leftrightarrow \begin{cases} z_1 = 0 \\ z_2 = 0 \\ w_1 = 0 \\ w_2 = 0 \end{cases}$$

$$\Leftrightarrow \quad z = w = 0_{\mathcal{M}_{\mathbb{H}}}.$$

Hence, $\ker\Phi = \{(0_{\mathcal{M}_{\mathbb{H}}}, 0_{\mathcal{M}_{\mathbb{H}}})\}$ so Φ is an isomorphism. Statement (ii) is a consequence of statement (i). Statement (iii) is also a consequence of statement (i) since any vector $\xi \in \mathbb{H}_{\mathbb{C}}$ can be written uniquely as the form $\xi = \begin{pmatrix} z_1 + iw_1 & z_2 + iw_2 \\ -\overline{z}_2 - i\overline{w}_2 & \overline{z}_1 + i\overline{w}_1 \end{pmatrix}$, $(z_1, w_1, z_2, w_2) \in \mathbb{C}^4$. Setting $z_k = a_k + ib_k$, $w_k = c_k + id_k$, $(a_k, b_k) \in \mathbb{R}^2$, $(c_k, d_k) \in \mathbb{R}^2$ for $k \in \{1, 2\}$, we have

$$\xi = \begin{pmatrix} a_1 - d_1 + i(b_1 + c_1) & a_2 - d_2 + i(b_2 + c_2) \\ -a_2 - d_2 + i(b_2 - c_2) & a_1 + d_1 - i(b_1 - c_1) \end{pmatrix}$$

$$= (a_1 + ic_1).\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (b_1 + id_1).\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$+(a_2 + ic_2).\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + (b_2 + id_2).\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Hence the family $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}$ generates the space $\mathbb{H}_{\mathbb{C}}$ and obviously is linearly independent. □

The following gives another specificity of the space $\mathbb{H}_{\mathbb{C}} \sim \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$.

Proposition 2.3. Each of the following statements holds:

(i) $(\mathbb{H}_{\mathbb{C}}, +, \times)$ *is a left* $\mathcal{M}_{\mathbb{H}}$*-vector space with basis* $\{\mathbb{1}, i\mathbb{1}\}$.

(ii) $(\mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}, +, \times)$ *is a left* $\mathcal{M}_{\mathbb{H}}$*-vector space with basis* $\{(\mathbb{1}, 0), (0, \mathbb{1})\}$.

Proof. Statement (i) holds since the following properties are satisfied:

(1) $\forall (z, w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}, \forall \xi \in \mathbb{H}_{\mathbb{C}}: \quad (z + w) \times \xi = z \times \xi + w \times \xi.$

(2) $\forall z \in \mathcal{M}_{\mathbb{H}}, \forall (\xi, \eta) \in \mathbb{H}_{\mathbb{C}} \times \mathbb{H}_{\mathbb{C}}: \quad z \times (\xi + \eta) = z \times \xi + z \times \eta.$

(3) $\forall (z, w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}, \forall \xi \in \mathbb{H}_{\mathbb{C}}: \quad (z \times w) \times \xi = z \times (w \times \xi).$

(4) $\forall \xi \in \mathbb{H}_{\mathbb{C}}, \quad \mathbb{1} \times \xi = \xi.$

Statement (ii) holds since the following properties are satisfied:

(1) $\forall (z, w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}, \forall (a, b) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}:$

$$(z + w) \times (a, b) = z \times (a, b) + w \times (a, b).$$

(2) $\forall z \in \mathcal{M}_{\mathbb{H}}, \forall ((a, b), (c, d)) \in (\mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}})^2:$

$$z \times [(a, b) + (c, d)] = z \times (a, b) + z \times (c, d).$$

(3) $\forall (z, w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}, \forall (a, b) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}:$

$$(z \times w) \times (a, b) = z \times (w \times (a, b)).$$

(4) $\forall (a, b) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}, \quad \mathbb{1} \times (a, b) = (a, b).$ □

Proposition 2.4. Let $\theta : \mathcal{M}_{\mathbb{H}} \to \mathbb{H}_{\mathbb{C}}$ *be the injection map and let* l *be an* $\mathbb{R}$*-linear map from* $\mathcal{M}_{\mathbb{H}}$ *to another* $\mathbb{C}$*-vector space* F*, then there is a unique* $\mathbb{C}$*-linear map* $\tilde{l}$ *from* $\mathbb{H}_{\mathbb{C}}$ *to* F *such that the following diagram*

$$\begin{array}{ccc} & \theta & \\ \mathcal{M}_{\mathbb{H}} & \rightarrow & \mathbb{H}_{\mathbb{C}} \\ & \searrow^{l} & \downarrow \tilde{l} \\ & & F \end{array}$$

commutes ($\tilde{l}\circ\theta = l$).

Proof. Let $\tilde{l}$ be the map such that

$$\tilde{l}(\xi) = \tilde{l}(z+iw) = l(z) + il(w) \quad \text{for all} \quad \xi = z + iw \in \mathbb{H}_{\mathbb{C}}.$$

Then, for all $\lambda = a + ib \in \mathbb{C}$, $(a, b) \in \mathbb{R}^2$, for all $\xi = z + iw \in \mathbb{H}_{\mathbb{C}}$, we have

$$\begin{aligned} \tilde{l}(\lambda.\xi) &= \tilde{l}[(az-bw)+i(bz+aw)] = l(az-bw) + il(bz+aw) \\ &= a(l(z)+il(w)) + ib(l(z)+il(w)) \\ &= (a+ib).[l(z)+il(w)] = (a+ib).\tilde{l}(z+iw) = \lambda.\tilde{l}(\xi). \end{aligned}$$

Hence the map $\tilde{l}$ is $\mathbb{C}$-linear. Further, $\tilde{l}$ is unique since the existence of two maps $\tilde{l}_1$ and $\tilde{l}_2$ such that $\tilde{l}_1\circ\theta = \tilde{l}_2\circ\theta = l$, provides that

$$\mathrm{Im}(\theta) \subset \ker\left(\tilde{l}_1 - \tilde{l}_2\right) \quad \text{and} \quad \mathrm{Im}(i\theta) \subset \ker\left(\tilde{l}_1 - \tilde{l}_2\right),$$

where $i\theta$ is the map defined by $i\theta(z) = iz$ for all $z \in \mathcal{M}_{\mathbb{H}}$. Which makes that $\ker(\tilde{l}_1 - \tilde{l}_2)$ is an $\mathbb{R}$-subspace of $\mathbb{H}_{\mathbb{C}}$, containing both $\mathcal{M}_{\mathbb{H}}$ and $i\mathcal{M}_{\mathbb{H}}$ so that $\ker(\tilde{l}_1 - \tilde{l}_2) = \mathcal{M}_{\mathbb{H}} + i\mathcal{M}_{\mathbb{H}} = \mathbb{H}_{\mathbb{C}}$. Thus $\tilde{l}_1 = \tilde{l}_2$. □

2.2 The conjugation in $\mathbb{H}_{\mathbb{C}}$

Let $\xi = z + iw \in \mathbb{H}_{\mathbb{C}}$ be a complex quaternion $(z, w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$. We define the conjugate of ξ to be the complex quaternion

$$\overline{\xi} = \overline{z} - i\overline{w}, \quad \left(\overline{z}, \overline{w}\right) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}.$$

Therefore, for all $(z_1, w_1, z_2, w_2) \in \mathbb{C}^4$, we have

$$\xi = \begin{pmatrix} z_1 + iw_1 & z_2 + iw_2 \\ -\overline{z}_2 - i\overline{w}_2 & \overline{z} + i\overline{w}_2 \end{pmatrix} \Leftrightarrow \overline{\xi} = \overline{z} - i\overline{w} = \begin{pmatrix} \overline{z}_1 - i\overline{w}_1 & \overline{z}_2 - i\overline{w}_2 \\ -z_2 + iw_2 & z - iw_2 \end{pmatrix}.$$

From now on, the notation $\mathrm{Tr}(\xi)$ stands for the trace of ξ, while ${}^t\xi$ stands for the transpose of ξ.

Proposition 2.5. For all $\xi = z + iw$ and all $\xi' = z' + iw'$ in $\mathbb{H}_{\mathbb{C}}$ with (z, w) and (z', w') in $\mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$, and for all $\lambda \in \mathbb{C}$, the conjugation map $\xi \mapsto \overline{\xi}$ satisfies on $\mathbb{H}_{\mathbb{C}}$ the following rules:

(i) $\overline{\overline{\xi}} = \xi$, $\quad \overline{\xi \pm \xi'} = \overline{\xi} \pm \overline{\xi'}$; $\quad \overline{\xi \times \xi'} = \overline{\xi} \times \overline{\xi'}$ *and* $\overline{\lambda.\xi} = \overline{\lambda}.\overline{\xi}$.

(ii) *The map $\xi \mapsto {}^t\xi$ is a $\mathbb{C}$-isomorphism of $\mathbb{H}_{\mathbb{C}}$ that satisfies $\overline{{}^t\xi} = {}^t\overline{\xi}$.*

(iii) *There exists a unique $(\xi_1, \xi_2) \in \mathbb{H}_{\mathbb{C}} \times \mathbb{H}_{\mathbb{C}}$ such that $\xi = \xi_1 + \xi_2$ with $\overline{\xi}_1 = i\xi_1$ and $\overline{\xi}_2 = -i\xi_2$.*

(iv) $\mathrm{Tr}(w \times {}^t\overline{z}) = \mathrm{Tr}(z \times {}^t\overline{w})$.

(v) $\xi \times {}^t\overline{\xi} = z \times {}^t\overline{z} + w \times {}^t\overline{w} + i(w \times {}^t\overline{z} - z \times {}^t\overline{w})$.

Proof. Statements (i) and (ii) are obvious since they can be proved by a direct computation. Let us prove statement (iii). The map $\xi \mapsto \overline{\xi}$, on $\mathbb{H}_{\mathbb{C}}$, induces on $\mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$, an $\mathbb{R}$-linear mapping J defined by $J(z, w) = (-w, z)$ and satisfies $J^2 = -Id$. Thus, J has two eigenvalues $\{-i, +i\}$. Let H^- be the eigenspace corresponding to the eigenvalue $-i$ and let H^+ be the eigenspace corresponding to the eigenvalue $+i$, then $\mathbb{H}_{\mathbb{C}} = H^- \oplus H^+$ and hence any $\xi = z + iw \in \mathbb{H}_{\mathbb{C}}$ can be written as $\xi = \xi_1 + \xi_2$ where $\xi_1 \in H^-$ and $\xi_2 \in H^+$. Indeed, we may take $\xi_1 = \frac{z-\overline{w}}{2} + i\frac{-\overline{z}+w}{2}$ and $\xi_2 = \frac{z+\overline{w}}{2} + i\frac{\overline{z}+w}{2}$. Finally, statements (iv) and (v) can be obtained by simple computations. □

2.3 $\mathbb{H}_{\mathbb{C}}$ as an inner product space

Proposition 2.6. Let $\langle \cdot \rangle$ denote the map

$$\begin{array}{ccc} \mathbb{H}_{\mathbb{C}} \times \mathbb{H}_{\mathbb{C}} & \to & \mathbb{C} \\ (\xi, \eta) & \mapsto & \langle \xi, \eta \rangle = Tr\left(\xi \times {}^t\overline{\eta}\right). \end{array}$$

(i) *For all $\eta \in \mathbb{H}_{\mathbb{C}}$, the map $\xi \mapsto \langle \xi, \eta \rangle$ is $\mathbb{C}$-linear.*

(ii) *For all $(\xi, \eta) \in \mathbb{H}_{\mathbb{C}} \times \mathbb{H}_{\mathbb{C}}$, we have $\langle \xi, \eta \rangle = \overline{\langle \eta, \xi \rangle}$.*
Therefore, $(\mathbb{H}_{\mathbb{C}}; \langle , \rangle)$ is an inner product space.

Proof. Statement (i) holds since for all $\lambda \in \mathbb{C}$ and all $\xi_1, \xi_2, \eta \in \mathbb{H}_{\mathbb{C}}$, we have

$$\begin{aligned} \langle \lambda.\xi_1 + \xi_2, \eta \rangle &= Tr\left((\lambda.\xi_1 + \xi_2) \times {}^t\overline{\eta}\right) \\ &= Tr\left(\lambda.\xi_1 \times {}^t\overline{\eta}\right) + Tr\left(\xi_2 \times {}^t\overline{\eta}\right) \\ &= \lambda\langle \xi_1, \eta \rangle + \langle \xi_2, \eta \rangle. \end{aligned}$$

Statement (ii) holds since for all $\xi, \eta \in \mathbb{H}_{\mathbb{C}}$, we have

$$\begin{aligned} \langle \xi, \eta \rangle &= Tr\left(\xi \times {}^t\overline{\eta}\right) = Tr\left({}^t\left(\xi \times {}^t\overline{\eta}\right)\right) \\ &= Tr\left(\overline{\eta} \times {}^t\xi\right) = \overline{Tr\left(\eta \times {}^t\overline{\xi}\right)} \\ &= \overline{\langle \eta, \xi \rangle}. \end{aligned}$$

□

Proposition 2.7. In the inner product space $(\mathbb{H}_{\mathbb{C}}; \langle , \rangle)$, each of the following statements holds.

(i) *For all $(\xi, \eta) \in \mathbb{H}_{\mathbb{C}} \times \mathbb{H}_{\mathbb{C}}$, if $\xi = x + iv$ and $\eta = y + iw$ with $(x, v), (y, w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$, then we have*

$$\langle \xi, \eta \rangle = \langle x, y \rangle + \langle v, w \rangle + i(\langle v, y \rangle - \langle x, w \rangle). \tag{2.1}$$

(ii) *For all $(z, w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$, we have $\langle iz, iw \rangle = \langle z, w \rangle = \langle w, z \rangle \in \mathbb{R}$.*

(iii) *For all $\xi = z + iw = \begin{pmatrix} z_1 + iw_1 & z_2 + iw_2 \\ -\overline{z}_2 - \overline{w}_2 & \overline{z}_1 + i\overline{w}_1 \end{pmatrix} \in \mathbb{H}_{\mathbb{C}}$, we have*

$$\langle \xi, \xi \rangle = |z_1 + iw_1|^2 + |z_1 - iw_1|^2 + |z_2 + iw_2|^2 + |z_2 - iw_2|^2. \tag{2.2}$$

(iv) *For all* $\xi = z + iw \in \mathbb{H}_{\mathbb{C}}$, *with* $(z, w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$, *we have*

$$\langle \xi, \xi \rangle = \langle z, z \rangle + \langle w, w \rangle. \tag{2.3}$$

(v) *For all* $\xi = z + iw \in \mathbb{H}_{\mathbb{C}}$, *with* $(z, w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$, *we have*

$$\det\xi = \det(z + iw) = \det z - \det w + iTr\left(z^t\overline{w}\right). \tag{2.4}$$

Furthermore, if for all $\xi \in \mathbb{H}_{\mathbb{C}}$, we put $\|\xi\| = \sqrt{\langle \xi, \xi \rangle}$, then $(\mathbb{H}_{\mathbb{C}}; \|.\|)$ is a $\mathbb{C}$-normed vector space.

Proof. Formula (2.1) in statement (i) is a consequence of the equality

$$\begin{aligned} \langle \xi, \eta \rangle = Tr\left(\xi \times {}^t\overline{\eta}\right) &= Tr\left(x \times {}^t\overline{y} + v \times {}^t\overline{w}\right) + iTr\left(v \times {}^t\overline{y} - x \times {}^t\overline{w}\right) \\ &= \langle x, y \rangle + \langle v, w \rangle + i(\langle v, y \rangle - \langle x, w \rangle). \end{aligned}.$$

Statement (ii) holds since for $z = \begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 & w_2 \\ -\overline{w}_2 & \overline{w}_1 \end{pmatrix}$ in $\mathcal{M}_{\mathbb{H}}$, $(z_1, z_2, w_1, w_2) \in \mathbb{C}^4$, we have

$$\begin{aligned} \langle iz, iw \rangle &= Tr\left((iz) \times {}^t\overline{(iw)}\right) = -i^2 Tr\left(z \times {}^t\overline{w}\right) \\ &= \langle z, w \rangle = z_1\overline{w}_1 + w_1\overline{z}_1 + z_2\overline{w}_2 w_2\overline{z}_2 \\ &= \langle w, z \rangle = Tr\left(w \times {}^t\overline{z}\right). \end{aligned}$$

Formula (2.2) in statement (iii) can be obtained by a direct computation of $Tr(\xi \times {}^t\overline{\xi})$. Formula (2.3) in statement (iv) holds since we have

$$\begin{aligned} \langle \xi, \xi \rangle &= \langle z + iw, z + iw \rangle = \langle z, z \rangle + \langle z, iw \rangle + \langle iw, z \rangle + \langle iw, iw \rangle \\ &= \langle z, z \rangle + \langle w, w \rangle + i(\langle z, w \rangle - \langle w, z \rangle) = \langle z, z \rangle + \langle w, w \rangle. \end{aligned}$$

Formula (2.4) in Statement (v) holds since we have

$$\begin{aligned} \det\xi &= \det(z + iw) = \begin{vmatrix} z_1 + iw_1 & z_2 + iw_2 \\ -\overline{z}_2 - i\overline{w}_2 & \overline{z}_1 + i\overline{w}_1 \end{vmatrix} \\ &= (z_1 + iw_1)\left(\overline{z}_1 + i\overline{w}_1\right) + (z_2 + iw_2)\left(\overline{z}_2 + i\overline{w}_2\right) \\ &= |z_1|^2 + |z_2|^2 - |w_1|^2 - |w_2|^2 + i\left(z_1\overline{w}_1 + w_1\overline{z}_1 + z_2\overline{w}_2 + \overline{z}_2 w_2\right) \\ &= \det z - \det w + iTr\left(z^t\overline{w}\right). \end{aligned}$$

□

Remark 2.8. (1) Since $\forall\,(\xi,\eta)\in\mathbb{H}_{\mathbb{C}}\times\mathbb{H}_{\mathbb{C}},\ {}^{t}\overline{(\xi\times\eta)}={}^{t}\overline{\eta}\times{}^{t}\overline{\xi}$ and $\forall\,\xi=\begin{pmatrix} z_1+iw_1 & z_2+iw_2 \\ -\overline{z}_2-i\overline{w}_2 & \overline{z}_1+i\overline{w}_1\end{pmatrix}\in\mathbb{H}_{\mathbb{C}},\ \langle\xi,\xi\rangle=Tr(\xi\times{}^{t}\overline{\xi})$, formula (2.2) can be transformed to

$$Tr\left(\xi\times{}^{t}\overline{\xi}\right)=|z_1+iw_1|^2+|z_2+iw_2|^2+|z_1-iw_1|^2+|z_2-iw_2|^2 \tag{2.5}$$

and gives

$$\forall\,(\xi,\eta)\in\mathbb{H}_{\mathbb{C}}\times\mathbb{H}_{\mathbb{C}},\quad \|\xi\times\eta\|.\le\|\xi\|.\|\eta\|. \tag{2.6}$$

Indeed, let $\eta=\begin{pmatrix} z_1'+iw_1' & z_2'+iw_2' \\ -\overline{z}_2'-i\overline{w}_2' & \overline{z}_1'+i\overline{w}_1'\end{pmatrix}\in\mathbb{H}_{\mathbb{C}}$. In vertu of the associativity of multiplication in $\mathbb{H}_{\mathbb{C}}$, the trace proprieties and formula (2.5), we have

$$\begin{aligned} Tr\left((\xi\times\eta)^{t}\left(\overline{\xi\times\eta}\right)\right) &= Tr\left(\xi\times\left(\eta\times{}^{t}\overline{\eta}\right)\times{}^{t}\overline{\xi}\right)\\ &= Tr\left(\left({}^{t}\overline{\xi}\times\xi\right)\left(\eta\times{}^{t}\overline{\eta}\right)\right)\\ &= \left[|z_1'+iw_1'|^2+|z_2'+iw_2'|^2\right]\left[|z_1-iw_1|^2+|z_2-iw_2|^2\right]\\ &\quad+\left[|z_1+iw_1|^2+|z_2+iw_2|^2\right]\left[|z_1'-iw_1'|^2+|z_2'-iw_2'|^2\right]. \end{aligned}$$

On the other hand, again by (2.5), we have

$$\begin{aligned} \|\xi\|^2\|\eta\|^2 &= Tr\left(\xi\times{}^{t}\overline{\xi}\right)Tr\left(\eta\times{}^{t}\overline{\eta}\right)\\ &= \left[|z_1+iw_1|^2+|z_2+iw_2|^2+|z_1-iw_1|^2+|z_2-iw_2|^2\right]\\ &\quad\times\left[|z_1'+iw_1'|^2+|z_2'+iw_2'|^2+|z_1'-iw_1'|^2+|z_2'-iw_2'|^2\right]\\ &\ge\left[|z_1'+iw_1'|^2+|z_2'+iw_2'|^2\right]\left[|z_1-iw_1|^2+|z_2-iw_2|^2\right]\\ &\quad+\left[|z_1+iw_1|^2+|z_2+iw_2|^2\right]\left[|z_1'-iw_1'|^2+|z_2'-iw_2'|^2\right]\\ &= \|\xi\times\eta\|^2. \end{aligned}$$

(2) For all $(z,w)\in\mathcal{M}_{\mathbb{H}}\times\mathcal{M}_{\mathbb{H}}$, an easy computation provides

$$\det(z+w)=\det z+\det w+Tr\left(z\times{}^{t}\overline{w}\right). \tag{2.7}$$

(3) For $z = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 & w_2 \\ -\overline{w}_2 & \overline{w}_1 \end{pmatrix}$ in $\mathcal{M}_{\mathbb{H}}$, we have $Tr(z \times {}^t\overline{w}) = z_1\overline{w}_1 + z_2\overline{w}_2 + w_1\bar{z}_1 + w_2\bar{z}_2$. Then, by the Cauchy–Schwarz inequality we deduce that

$$|Tr(z \times {}^t\overline{w})|^2 \leq (\det z)(\det w). \tag{2.8}$$

Moreover, if $\xi = z + iw \in \mathbb{H}_{\mathbb{C}}$, then (2.4) provides that.

$$|\det(z + iw)| \leq \det z + \det w. \tag{2.9}$$

Indeed, following (2.4) and (2.8), we have

$$\begin{aligned} \left|\det(z + iw)\right|^2 &= \left|\det z - \det w + iTr\left(z \times {}^t\overline{w}\right)\right|^2 \\ &= (\det z - \det w)^2 + \left|Tr\left(z \times {}^t\overline{w}\right)\right|^2 \\ &\leq (\det z - \det w)^2 + (\det z)(\det w) \\ &\leq (\det z + \det w)^2. \end{aligned}$$

3. Functions of one complex quaternion variable and complex quaternionic differentiability

3.1 Functions of one complex quaternion variable

Let $E \in \{\mathbb{R}, \mathbb{C}, \mathcal{M}_{\mathbb{H}}, \mathbb{H}_{\mathbb{C}}\}$ and $\mathcal{D}$ be an open subset in $\mathbb{H}_{\mathbb{C}}$. We say that

$$f: \begin{array}{ccc} \mathcal{D} & \to & E \\ \xi & \mapsto & f(\xi) \end{array}$$

is a function of one complex quaternion variable $\xi \in \mathbb{H}_{\mathbb{C}}$ (or a complex quaternionic function or a complex matrix function), if f is an association which associates to each element $\xi \in \mathcal{D}$ an element $f(\xi) \in E$. In case $E = \mathbb{H}_{\mathbb{C}}$, which means that $f(\xi) \in \mathbb{H}_{\mathbb{C}}$ for all $\xi \in \mathbb{H}_{\mathbb{C}}$, we deduce that for all $(z, w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$, there exist $g_1(z, w) = g_1(z, \bar{z}, w, \overline{w}) \in \mathcal{M}_{\mathbb{H}}$ and $g_2(z, w) = g_2(z, \bar{z}, w, \overline{w}) \in \mathcal{M}_{\mathbb{H}}$ such that

$$\begin{aligned} f(\xi) &= g_1(\xi) + ig_2(\xi) \\ &= g_1(z + iw) + ig_2(z + iw) \\ &= g_1(z, w) + ig_2(z, w) \\ &= g_1\left(z, \bar{z}, w, \overline{w}\right) + ig_2\left(z, \bar{z}, w, \overline{w}\right). \end{aligned}$$

3.2 Quaternionic $\mathbb{R}$-differentiability

Definition 3.1. For $E \in \{\mathbb{R}, \mathbb{C}, \mathcal{M}_{\mathbb{H}}, \mathbb{H}_{\mathbb{C}}\}$, let $f: \mathcal{D} \to E$ be a complex quaternionic function defined on an open subset $\mathcal{D}$ of $\mathbb{H}_{\mathbb{C}}$.

(i) We say that f is quaternionic $\mathbb{R}$-differentiable (or simply $\mathbb{R}$-differentiable) at point $\xi_0 \in \mathcal{D}$, if there exists an $\mathbb{R}$-linear map $f'(\xi_0): \mathbb{H}_{\mathbb{C}} \to E$ such that

$$\lim_{h \to 0} \frac{1}{||h||}\left[f(\xi_0 + h) - f(\xi_0) - f'(\xi_0)(h)\right] = 0.$$

Which is equivalent to $$\lim_{h\to 0}\frac{\left\|f(\xi_0+h)-f(\xi_0)-f'(\xi_0)(h)\right\|}{\|h\|}=0.$$

(ii) We say that f is $\mathbb{R}$-differentiable on $\mathcal{D}$ if it is $\mathbb{R}$-differentiable at any point $\xi\in\mathcal{D}$.

Example 3.2. Let us illustrate Definition 3.1 by the following examples:

(1) The function $f:\begin{array}{ccc}\mathbb{H}_{\mathbb{C}} & \to & \mathbb{R}\\ \xi & \mapsto & |\xi|^2\end{array}$ is $\mathbb{R}$-differentiable and for all $h\in\mathbb{H}_{\mathbb{C}}$, we have
$$f'(\xi)(h)=Tr\left(h\times{}^t\overline{\xi}+\xi\times{}^t\overline{h}\right).$$

In particular, at point $h=\mathbb{1}$, we have $f'(\xi)(\mathbb{1})=Tr\left(\xi+{}^t\overline{\xi}\right)$. Indeed, for all $h\in\mathbb{H}_{\mathbb{C}}$, we have
$$\|\xi+h\|^2=\|\xi\|^2+2\Re\langle h,\xi\rangle+\|h\|^2.$$

Further, the map $h\mapsto 2\Re\langle h,\xi\rangle=Tr\left(h\times{}^t\overline{\xi}+\xi\times{}^t\overline{h}\right)$ is $\mathbb{R}$-linear on $\mathbb{H}_{\mathbb{C}}$ and the function $h\mapsto\varepsilon(h)=\|h\|^2$ satisfies $\lim_{h\to 0}\frac{\varepsilon(h)}{\|h\|}=0$, so that, f is $\mathbb{R}$-differentiable.

(2) The function $f:\begin{array}{ccc}\mathbb{H}_{\mathbb{C}} & \to & \mathbb{C}\\ \xi & \mapsto & \det(\xi)\end{array}$ is $\mathbb{R}$-differentiable and its differential is defined by
$$\forall h=h_1+ih_2\in\mathbb{H}_{\mathbb{C}},\quad f'(\xi)(h)=\langle\xi,\tilde{h}\rangle,$$
where $\tilde{h}=h_1-ih_2,(h_1,h_2)\in(\mathcal{M}_{\mathbb{H}})^2$. In particular, for all $\xi\in\mathbb{H}_{\mathbb{C}}$, at point $h=\mathbb{1}$, $f'(\xi)(\mathbb{1})=\langle\xi,\mathbb{1}\rangle=Tr(\xi)$. Indeed, by formula (2.4) in Proposition 2.7, for all $h=h_1+ih_2\in\mathbb{H}_{\mathbb{C}}$ and for all $\xi=z+iw\in\mathbb{H}_{\mathbb{C}}$, we have
$$\begin{aligned}\det(\xi+h)&=\det(z+h_1+i(w+h_2))\\&=\det\xi+\det h+\langle\xi,\tilde{h}\rangle\end{aligned}$$
where $\langle\xi,\tilde{h}\rangle=Tr\left(z\times{}^t\overline{h}_1-w\times{}^t\overline{h}_2+i(h_1\times{}^t\overline{w}+z\times{}^t\overline{h}_2)\right)$.

It is clear that
$$h\mapsto\langle\xi,\tilde{h}\rangle=Tr\left(\xi\times{}^t\overline{\tilde{h}}\right)$$
is an $\mathbb{R}$-linear map on $\mathbb{H}_{\mathbb{C}}$. In addition, due to (2.1), the function $h\mapsto\varepsilon(h)=\det h$ satisfies $\varepsilon(h)=o(\|h\|)$ since $\lim_{h\to 0}\frac{\varepsilon(h)}{\|h\|}=0$. In fact, by (2.2) in Proposition 2.7, we know that $\|h\|^2=|h_1|^2+|h_2|^2$. Moreover, by (2.9) given in Remark 2.8, for $\|h\|\neq 0$, we have
$$\begin{aligned}\frac{|\det(h)|^2}{\|h\|^2}&=\frac{|\det(h_1+ih_2)|^2}{\|h\|^2}\\&\leq\frac{(\det h_1+\det h_2)^2}{\|h\|^2}\\&=\frac{\left(|h_1|^2+|h_2|^2\right)^2}{\|h\|^2}\\&=\|h\|^2.\end{aligned}$$

Which affirms that $\lim_{h\to 0}\frac{\varepsilon(h)}{\|h\|}=0$.

3.3 Quaternionic directional derivative

Definition 3.3. For $E \in \{\mathbb{R}, \mathbb{C}, \mathcal{M}_{\mathbb{H}}, \mathbb{H}_{\mathbb{C}}\}$, let

$$f : \begin{array}{ccc} \mathcal{D} & \rightarrow & E \\ z + iw & \mapsto & f(z + iw) \end{array}$$

be a complex quaternionic function of one variable $\xi = z + iw \in \mathbb{H}_{\mathbb{C}}$ defined on an open subset $\mathcal{D}$ of $\mathbb{H}_{\mathbb{C}}$. If $\eta \in \mathbb{H}_{\mathbb{C}} \setminus \{0\}$ is a nonzero complex quaternion, then we say that f has a quaternionic directional derivative at point $\xi_0 \in \mathcal{D}$, in the direction of the vector η, if the function of one real variable $t \in \mathbb{R} \mapsto f(\xi_0 + t\eta)$ is differentiable at point 0. We denote $f'_\eta(\xi_0)$ the derivative of f at point ξ_0 in the direction of η. Hence,

$$f'_\eta(\xi_0) = \lim_{t \to 0} \frac{1}{t} [f(\xi_0 + t\eta) - f(\xi_0)].$$

3.4 Quaternionic partial derivative

As we have $\mathbb{H}_{\mathbb{C}} \sim \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$, we may suppose for the present, that $\mathcal{D}$ is an open subset of $\mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$. Let $E \in \{\mathbb{R}, \mathbb{C}, \mathcal{M}_{\mathbb{H}}, \mathbb{H}_{\mathbb{C}}\}$ and

$$f : \begin{array}{ccc} \mathcal{D} & \rightarrow & E \\ (z, w) & \mapsto & f(z, w) \end{array}$$

be a complex quaternionic function defined on $\mathcal{D}$ such that

$$f(z, w) = f\left(z, \bar{z}, w, \overline{w}\right), \quad \forall (z, w) \in \mathcal{D}.$$

We say that f has a quaternionic (or matrix) partial derivative $\frac{\partial f}{\partial z}(z, w)$ with respect to the variable z, if it is differentiable in the direction of the vector $(\mathbb{1}, 0)$ and we say that f has a quaternionic partial derivative $\frac{\partial f}{\partial w}(z, w)$ with respect to the variable w, if it is differentiable in the direction of the vector $(0, \mathbb{1})$. Therefore, we have

$$\begin{aligned} \frac{\partial f}{\partial z}(z, w) &= \lim_{h \to 0, h \in \mathbb{R} \setminus \{0\}} \frac{1}{h} [f(z + h\mathbb{1}, w) - f(z, w)], \\ \frac{\partial f}{\partial w}(z, w) &= \lim_{h \to 0, h \in \mathbb{R} \setminus \{0\}} \frac{1}{h} [f(z, w + h\mathbb{1}) - f(z, w)]. \end{aligned}$$

Similarly, if $(\bar{z}, \overline{w}) \in \mathcal{D}$ for all $(z, w) \in \mathcal{D}$, then we say that f has a quaternionic partial derivative $\frac{\partial f}{\partial \bar{z}}(z, w)$ with respect to the variable $\bar{z}$, if the function

$$\tilde{f} : \begin{array}{ccc} \mathcal{D} & \rightarrow & E \\ \left(\bar{z}, \overline{w}\right) & \mapsto & \tilde{f}\left(\bar{z}, \overline{w}\right) = f\left(z, \bar{z}, w, \overline{w}\right) \end{array}$$

is differentiable in the direction of the vector $(\mathbb{1}, 0)$ and we say that f has a partial derivative $\frac{\partial f}{\partial \overline{w}}(z, w)$ with respect to the variable $\overline{w}$, if the function

$$\tilde{f} : \begin{array}{ccc} \mathcal{D} & \rightarrow & E \\ \left(\bar{z}, \overline{w}\right) & \mapsto & \tilde{f}\left(\bar{z}, \overline{w}\right) = f\left(z, \bar{z}, w, \overline{w}\right) \end{array}$$

is differentiable in the direction of the vector $(0, \mathbb{1})$. Therefore, we have

$$\frac{\partial f}{\partial \bar{z}}(z,w) = \lim_{h\to 0, h\in\mathbb{R}\setminus\{0\}} \frac{1}{h}\left[f\left(z,\bar{z}+h\mathbb{1},w,\overline{w}\right) - f\left(z,\bar{z},w,\overline{w}\right)\right]$$

$$\frac{\partial f}{\partial \overline{w}}(z,w) = \lim_{h\to 0, h\in\mathbb{R}\setminus\{0\}} \frac{1}{h}\left[f\left(z,\bar{z},w,\overline{w}+h\mathbb{1}\right) - f\left(z,\bar{z},w,\overline{w}\right)\right].$$

Example 3.4. (1) The function $f: \begin{array}{ccc} \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}} & \to & \mathcal{M}_{\mathbb{H}} \\ (z,w) & \mapsto & (z+w)^2 \end{array}$ has partial derivatives such that $\frac{\partial f}{\partial z}(z,w) = \frac{\partial f}{\partial w}(z,w) = 2(z+w)$.

(2) The function $f: \begin{array}{ccc} \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}} & \to & \mathcal{M}_{\mathbb{H}} \\ (z,w) & \mapsto & {}^{t}(z+\overline{w}) \end{array}$ has partial derivatives such that $\frac{\partial f}{\partial z}(z,w) = \frac{\partial f}{\partial \overline{w}}(z,w) = \mathbb{1}$.

(3) The function $f: \begin{array}{ccc} \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}} & \to & \mathbb{C} \\ (z,w) & \mapsto & Tr(z^t\overline{w}) \end{array}$ satisfies the fact that $f(\bar{z},\overline{w}) = f(z,w) = f(w,z)$ and an easy computation gives

$$\frac{\partial f}{\partial \bar{z}}(z,w) = \frac{\partial f}{\partial z}(z,w) = Tr\left(\overline{w}\right) = Tr(w)$$

$$\frac{\partial f}{\partial w}(z,w) = \frac{\partial f}{\partial \overline{w}}(z,w) = Tr\left(\bar{z}\right) = Tr(z).$$

(4) The function $f: \begin{array}{ccc} \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}} & \to & \mathbb{C} \\ (z,w) & \mapsto & Tr(z+w) \end{array}$ satisfies the fact that $f(\bar{z},\overline{w}) = f(z,w) = f(w,z)$ and an easy computation gives

$$\frac{\partial f}{\partial \bar{z}}(z,w) = \frac{\partial f}{\partial z}(z,w) = Tr(\mathbb{1})$$

$$\frac{\partial f}{\partial \overline{w}}(z,w) = \frac{\partial f}{\partial w}(z,w) = Tr(\mathbb{1}).$$

(5) Using formula (2.7) in Remark 2.8, expressing that

$$\forall (z,w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}, \quad \det(z+w) = \det z + \det w + Tr\left(z^t\overline{w}\right).$$

Hence, the partial derivatives of $f: \begin{array}{ccc} \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}} & \to & \mathbb{R}_{+} \\ (z,w) & \mapsto & \det(z+w) \end{array}$ are such that $\frac{\partial f}{\partial z}(z,w) = \frac{\partial f}{\partial w}(z,w) = Tr(z+w)$.

3.5 Complex quaternionic derivative

For $E \in \{\mathbb{R}, \mathbb{C}, \mathcal{M}_{\mathbb{H}}, \mathbb{H}_{\mathbb{C}}\}$, let $f: \mathcal{D} \to E$ be a complex quaternionic function defined on an open subset $\mathcal{D}$ of $\mathbb{H}_{\mathbb{C}}$. We shall define what means by a quaternionic $\mathbb{C}$-differentiable function of one complex quaternion variable.

Definition 3.5. (i). We say that f has a complex quaternionic derivative (or complex matrix derivative) at point $\xi_0 \in \mathcal{D}$, if there exists a $\mathbb{C}$-linear map $f'(\xi_0) : \mathbb{H}_{\mathbb{C}} \to E$ such that

$$\lim_{h\to 0}\frac{1}{\|h\|}\left[f(\xi_0+h)-f(\xi_0)-f'(\xi_0)(h)\right]=0.$$

which means that

$$\lim_{h\to 0}\frac{\left\|f(\xi_0+h)-f(\xi_0)-f'(\xi_0)(h)\right\|}{\|h\|}=0.$$

(ii). We say that f is complex quaternionic differentiable on $\mathcal{D}$ if it has a complex quaternionic derivative at any point $\xi \in \mathcal{D}$. In that case f is said to be quaternionic holomorphic (or matrix holomorphic) on $\mathcal{D}$.

Remark 3.6. Statement (i) in Definition 3.5 can be written as follows. The function f has a complex quaternionic derivative at point $\xi_0 \in \mathcal{D}$, if there exists a $\mathbb{C}$-linear map $f'(\xi_0) : \mathbb{H}_{\mathbb{C}} \to E$ and a function $h \mapsto \varepsilon(h)$ defined on a neighborhood $\mathcal{V}$ of 0 satisfying $\lim_{h\to 0}\varepsilon(h) = 0$ and such that for all $h \in \mathcal{V}$, we have $f(\xi_0 + h) = f(\xi_0) + f'(\xi_0)(h) + \|h\|\varepsilon(h)$. Moreover, a complex quaternionic differentiable function is obviously continuous.

Example 3.7. (1) The function $f : \begin{array}{ccc}\mathbb{H}_{\mathbb{C}} & \to & \mathbb{R}\\ \xi & \mapsto & \|\xi\|^2\end{array}$ is not holomorphic on $\mathbb{H}_{\mathbb{C}}$ since it is complex differentiable only at $\xi = 0$. Indeed,

$$\forall \xi \in \mathbb{H}_{\mathbb{C}}, \quad \forall\, h \in \mathbb{H}_{\mathbb{C}}, \quad \|\xi+h\|^2 = \|\xi\|^2 + 2\Re\langle h, \xi\rangle + \|h\|^2.$$

Moreover, $h \mapsto \varepsilon(h) = \|h\|^2$ satisfies $\lim_{h\to 0}\frac{\varepsilon(h)}{\|h\|} = 0$, but, for all $\xi \in \mathbb{H}_{\mathbb{C}} \setminus \{0\}$, the map $h \mapsto 2\Re\langle h, \xi\rangle = Tr(h^t\overline{\xi} + \xi^t\overline{h})$ is not $\mathbb{C}$-linear.

(2) If $S = \{\xi \in \mathbb{H}_{\mathbb{C}} : \det\xi = 0\}$, then $\mathbb{H}_{\mathbb{C}} \setminus S$ is an open subset of $\mathbb{H}_{\mathbb{C}}$ since the function $\xi \mapsto \det\xi$ is continuous on $\mathbb{H}_{\mathbb{C}}$. Let us prove that the function $f : \begin{array}{ccc}\mathbb{H}_{\mathbb{C}}\setminus S & \to & \mathbb{H}_{\mathbb{C}}\setminus S\\ \xi & \mapsto & \xi^{-1}.\end{array}$ is $\mathbb{C}$-differentiable on $\mathbb{H}_{\mathbb{C}} \setminus S$ and $f'(\xi)$ is such that

$$\forall\, \xi \in \mathbb{H}_{\mathbb{C}} \setminus S,\ \forall\, h \in \mathbb{H}_{\mathbb{C}}, \quad f'(\xi)(h) = -\xi^{-1}h\xi^{-1}.$$

In particular, at point $h = \mathbb{1}$, we have $f'(\xi)(\mathbb{1}) = -\xi^{-2}$. So that f is holomorphic on $\mathbb{H}_{\mathbb{C}} \setminus S$. Indeed, following Example 3.2, $\forall\, \xi \in \mathbb{H}_{\mathbb{C}}$ and $\forall\, h = h_1 + ih_2 \in \mathbb{H}_{\mathbb{C}}$, we have

$$\det(\xi + h) = \det\xi + \det h + \left\langle \xi, \tilde{h}\right\rangle, \quad \text{where} \quad \tilde{h} = h_1 - ih_2.$$

By the Cauchy–Schwarz inequality and the Example 3.2 (2), we have

$$\left|\left\langle \xi, \tilde{h}\right\rangle\right| \le \|\xi\|\|h\| \quad \text{and} \quad |\det h| \le \|h\|^2.$$

So that, if $\|h\|$ is small enough, then $|\det(h)|$ and $|\langle \xi, \tilde{h}\rangle|$ are also small enough. Hence the sum $(\xi + h)$ is invertible in $\mathbb{H}_{\mathbb{C}}$ whenever ξ is invertible in $\mathbb{H}_{\mathbb{C}}$ and h is small enough. Furthermore, for all $\xi \in \mathbb{H}_{\mathbb{C}} \setminus S$ and all $h \in \mathbb{H}_{\mathbb{C}}$ we have

$$(\xi + h)\left(\xi^{-1} - \xi^{-1}h\xi^{-1}\right) = \mathbb{1} - h\xi^{-1}h\xi^{-1}.$$

Therefore, $\forall \xi \in \mathbb{H}_{\mathbb{C}} \setminus S$ and $\forall\, h \in \mathbb{H}_{\mathbb{C}}$ small enough,

$$(\xi + h)^{-1} = \xi^{-1} - \xi^{-1}h\xi^{-1} + (\xi + h)^{-1}h\xi^{-1}h\xi^{-1}.$$

Let $\varepsilon(h) = (\xi + h)^{-1}h\xi^{-1}h\xi^{-1}$, since $h \mapsto \xi^{-1}h\xi^{-1}$ is a $\mathbb{C}$-linear map on $\mathbb{H}_{\mathbb{C}}$, to prove that f is $\mathbb{C}$-differentiable at point ξ, it is sufficient to prove that $\varepsilon(h) = o(\|h\|)$. On the other hand, using inequality (2.6) in Remark 2.8 we get:

$$\begin{aligned}\|\varepsilon(h)\| = \|(\xi + h)^{-1}h\xi^{-1}h\xi^{-1}\| \quad &\leq \quad \|(\xi + h)^{-1}\|\|h\|\|\xi^{-1}\|\|h\|\|\xi^{-1}\| \\ &\leq \quad \|(\xi + h)^{-1}\|\|\xi^{-1}\|^2\|h\|^2.\end{aligned}$$

hence, $\lim_{h\to 0}\frac{|\varepsilon(h)|}{|h|} = 0$, and so f is complex differentiable on $\mathbb{H}_{\mathbb{C}} \setminus S$.

3.6 Operations on complex quaternionic differentiable functions

Proposition 3.8. Let $E \in \{\mathbb{R}, \mathbb{C}, \mathcal{M}_{\mathbb{H}}, \mathbb{H}_{\mathbb{C}}\}$, $f : \mathcal{D} \to E$ *and* $g : \mathcal{D} \to E$ *be two complex quaternionic functions defined on the open subset* $\mathcal{D}$ *of* $\mathbb{H}_{\mathbb{C}}$ *and let* $\lambda \in \mathbb{C}$.

(i) *If f and g are both complex differentiable at $\xi \in \mathcal{D}$, then so is the function $f + \lambda g$ and for all $h \in \mathbb{H}_{\mathbb{C}}$, we have*

$$(f + \lambda g)'(\xi)(h) = f'(\xi)(h) + \lambda g'(\xi)(h).$$

(ii) *If f and g are both complex differentiable at $\xi \in \mathcal{D}$, then so are the functions $f.g$ and $g.f$. Moreover, for all $h \in \mathbb{H}_{\mathbb{C}}$, we have*

$$(f.g)'(\xi)(h) = f'(\xi)(h).g(\xi) + f(\xi).g'(\xi)(h).$$

Proof. Statement (i) is obvious. Let us justify statement (ii). Since f and g are both $\mathbb{C}$-differentiable at point ξ, we have

$$\begin{aligned} f(\xi + h) \quad &= f(\xi) + f'(\xi)(h) + |h|\varepsilon_1(h), \quad \lim_{h\to 0}\varepsilon_1(h) = 0. \\ g(\xi + h) \quad &= g(\xi) + g'(\xi)(h) + |h|\varepsilon_2(h), \quad \lim_{h\to 0}\varepsilon_2(h) = 0.\end{aligned}$$

Computing the product of the above equalities, gives that

$$f.g(\xi + h) = f.g(\xi) + f'(\xi)(h).g(\xi) + f(\xi).g'(\xi)(h) + |h|\varepsilon_3(h), \quad \lim_{h\to 0}\varepsilon_3(h) = 0.$$

□

Corollary 3.9. Any quaternionic polynomial function $f(\xi) = \sum_{n=0}^{d} a_n\xi^n$ *of degree* $d \geq 1$, *with coefficients in* $\mathbb{H}_{\mathbb{C}}$, *is* $\mathbb{C}$-*differentiable and its complex quaternionic derivative is given at any* $\xi \in \mathbb{H}_{\mathbb{C}}$ *by*

$$\forall\, h \in \mathbb{H}_{\mathbb{C}}, \quad f'(\xi)(h) = \sum_{n=1}^{d-1} a_n\left(\sum_{k=0}^{n-1} \xi^k h\xi^{n-k-1}\right). \tag{3.1}$$

Proof. By Proposition 3.8, it suffices to prove the result for a quaternionic monomial $f(\xi) = \xi^n$, which can be proved by induction on $n \in \mathbb{N}$. First, let show the existence of a function $h \mapsto \varepsilon(h)$ such that for all $\xi \in \mathbb{H}_{\mathbb{C}}$ and for all $h \in \mathbb{H}_{\mathbb{C}}$, we have

$$(\xi + h)^n = \xi^n + \sum_{k=0}^{n-1} \xi^k h \xi^{n-k-1} + \varepsilon(h) \quad \text{and} \quad \varepsilon(h) = o(|h|).$$

This is obvious for $n = 1$, (we take $\varepsilon(h) = 0$). Suppose the statement holds for all $k \in \mathbb{N}, 1 \leq k \leq n$ where $n > 1$ is a natural number. Then we have

$$\begin{aligned}
(\xi + h)^{n+1} &= (\xi + h)\left[\xi^n + \sum_{k=0}^{n-1} \xi^k h \xi^{n-k-1} + \varepsilon(h)\right] \\
&= \xi^{n+1} + (\xi + h)\sum_{k=0}^{n-1} \xi^k h \xi^{n-k-1} + h\xi^n + (\xi + h)\varepsilon(h) \\
&= \xi^{n+1} + \sum_{k=0}^{n-1} \xi^{k+1} h \xi^{n-k-1} + h\xi^n + h\sum_{k=1}^{n-1} \xi^k h \xi^{n-k-1} \\
&\quad + (\xi + h)\varepsilon(h) \\
&= \xi^{n+1} + \sum_{k=0}^{(n+1)-1} \xi^k h \xi^{(n+1)-k-1} + \varepsilon_1(h)
\end{aligned}$$

where

$$\varepsilon_1(h) = h\sum_{k=1}^{n-1} \xi^k h \xi^{n-k-1} + (\xi + h)\varepsilon(h).$$

Since the map defined on $\mathbb{H}_{\mathbb{C}}$ by

$$h \mapsto \sum_{k=0}^{n-1} \xi^k h \xi^{n-k-1}$$

is $\mathbb{C}$-linear and since $\lim_{h\to 0} \frac{\varepsilon_1(h)}{\|h\|} = 0$. Then, f is holomorphic on $\mathbb{H}_{\mathbb{C}}$. □

Remark 3.10. (1) If $a \in \mathbb{H}_{\mathbb{C}} \setminus S$ $(\det a \neq 0)$ and $b \in \mathbb{H}_{\mathbb{C}}$, then the quaternionic polynomial function $f(\xi) = a\xi + b$ of degree 1, is holomorphic on $\mathbb{H}_{\mathbb{C}}$ and $\xi_0 = -a^{-1}b$ is its unique zero.

(2) At point $h = \mathbb{1}$, formula (3.1) provides $f'(\xi)(\mathbb{1}) = \sum_{n=1}^{d-1} n a_n \xi^{n-1}$.

Proposition 3.11. Let $\mathcal{D}$ and $\mathcal{D}'$ be two open subsets of $\mathbb{H}_{\mathbb{C}}$ and let f and g be complex quaternionic functions defined on $\mathcal{D}$ and $\mathcal{D}'$ respectively. Suppose that $f(\mathcal{D}) \subset \mathcal{D}'$, f is complex quaternionic differentiable at $\xi \in \mathcal{D}$ and g is complex quaternionic differentiable at $f(\xi)$, then the composite function $g \circ f$ is complex quaternionic differentiable at ξ and its complex quaternion derivative is given by

$$\forall\, h \in \mathbb{H}_{\mathbb{C}}, \quad (g \circ f)'(\xi)(h) = [g'(f(\xi)) \circ f'(\xi)](h).$$

Proof. Since f is $\mathbb{C}$-differentiable at ξ, then for all $h \in \mathbb{H}_{\mathbb{C}}$ we have

$$f(\xi + h) = f(\xi) + f'(\xi)(h) + \|h\|\varepsilon_1(h), \quad \lim_{h\to 0} \varepsilon_1(h) = 0,$$

and since g is $\mathbb{C}$-differentiable at $f(\xi)$, for all $k \in \mathbb{H}_{\mathbb{C}}$, we have

$$g(f(\xi)+k) = g(f(\xi))) + g'(f(\xi))(k) + \|k\|\varepsilon_2(k), \quad \lim_{k\to 0}\varepsilon_2(k) = 0.$$

Hence by composition we find that

$$\begin{aligned} g\circ f(\xi+h) &= g(f(\xi+h)) = g\big(f(\xi)+f'(\xi)(h)+\|h\|\varepsilon_1(h)\big) \\ &= g(f(\xi))) + g'(f(\xi)) \times \big(f'(\xi)(h)\big) + \|h\|\varepsilon_3(h) \\ &= g(f(\xi))) + g'(f(\xi))\circ f'(\xi)(h) + \|h\|\varepsilon_3(h) \end{aligned}$$

where, $\lim_{h\to 0}\varepsilon_3(h) = 0$. Thus, $g\circ f$ is $\mathbb{C}$-differentiable at ξ. □

4. Quaternionic holomorphic structure on $\mathbb{H}_{\mathbb{C}}$

Let ∂_ξ and $\overline{\partial}_\xi$ be the operators defined on the space of differentiable quaternionic functions of one variable $\xi \in \mathbb{H}_{\mathbb{C}}$ by

$$\partial_\xi =: \partial = \frac{\partial}{\partial z} - i\frac{\partial}{\partial w} \quad \text{and} \quad \overline{\partial}_\xi =: \overline{\partial} = \frac{\partial}{\partial \overline{z}} - i\frac{\partial}{\partial \overline{w}}.$$

Let Φ be a complex quaternionic differentiable (complex matrix differentiable) function of one variable $\xi = z + iw \in \mathbb{H}_{\mathbb{C}}$ where $(z,w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$. Then Φ can be written $\Phi(z+iw) = f(z, w) + ig(z, w)$ and we get

$$\begin{aligned} \partial(\Phi) &= \frac{\partial\Phi}{\partial\xi} = \left(\frac{\partial}{\partial z} - i\frac{\partial}{\partial w}\right)(f+ig) \\ &= \left(\frac{\partial f}{\partial z} - i\frac{\partial f}{\partial w}\right) + i\left(\frac{\partial g}{\partial z} - i\frac{\partial g}{\partial w}\right) \\ &= \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial w}\right) + i\left(\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\right) \\ \overline{\partial}(\Phi) &= \frac{\partial\Phi}{\partial\overline{\xi}} = \left(\frac{\partial}{\partial \overline{z}} - i\frac{\partial}{\partial \overline{w}}\right)(f+ig) \\ &= \left(\frac{\partial f}{\partial \overline{z}} - i\frac{\partial f}{\partial \overline{w}}\right) - i\left(\frac{\partial g}{\partial \overline{z}} - i\frac{\partial g}{\partial \overline{w}}\right) \\ &= \left(\frac{\partial f}{\partial \overline{z}} - \frac{\partial g}{\partial \overline{w}}\right) - i\left(\frac{\partial f}{\partial \overline{w}} + \frac{\partial g}{\partial \overline{z}}\right). \end{aligned}$$

4.1 Cauchy–Riemann quaternionic differential equations

The following criterion provides a necessary and sufficient condition for the holomorphicity of complex quaternionic functions.

Theorem 4.1. *Let $E \in \{\mathbb{R}, \mathbb{C}, \mathcal{M}_{\mathbb{H}}, \mathbb{H}_{\mathbb{C}}\}$ and $\Phi : \mathcal{D} \to E$ be a complex quaternionic function of one complex quaternion variable $\xi = z + iw$ with $(z,w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$, defined on an open subset $\mathcal{D}$ in $\mathbb{H}_{\mathbb{C}}$. Suppose that*

$$\Phi(\xi) = \Phi(z+iw) = f(z,w) + ig(z,w),$$

then, Φ is holomorphic on $\mathcal{D}$, if and only if the following Cauchy–Riemann type equations $\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \overline{w}}$ and $\frac{\partial f}{\partial \overline{w}} = -\frac{\partial g}{\partial \bar{z}}$ are satisfied.

Proof. Since the spaces $\mathbb{H}_{\mathbb{C}}$ and its dual $T_{\mathbb{H}_{\mathbb{C}}}$ are isomorphic as two dimensional $\mathcal{M}_{\mathbb{H}}$-vector spaces, then the family $\left\{\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \bar{\xi}}\right\}$ constitutes a basis of the tangent space $T_{\mathbb{H}_{\mathbb{C}}}$ over $\mathcal{M}_{\mathbb{H}}$. Therefore, the function Φ is holomorphic on $\mathcal{D}$, if and only if for all $\xi \in \mathcal{D}$, the map $\Phi'(\xi)$ is $\mathbb{C}$-linear. Since $\Phi'(\xi)$ can be written in the basis $\left\{\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \bar{\xi}}\right\}$, then the $\mathbb{C}$-linearity condition of the map $\Phi'(\xi)$ is equivalent to $\overline{\partial}_{\xi}\Phi = 0$ which is equivalent to the following

$$\left(\frac{\partial f}{\partial \bar{z}} - \frac{\partial g}{\partial \overline{w}}\right) - i\left(\frac{\partial f}{\partial \overline{w}} + \frac{\partial g}{\partial \bar{z}}\right) = 0$$

$$\Leftrightarrow$$

$$\left(\frac{\partial f}{\partial \bar{z}} - \frac{\partial g}{\partial \overline{w}}\right) = \left(\frac{\partial f}{\partial \overline{w}} + \frac{\partial g}{\partial \bar{z}}\right) = 0$$

$$\Leftrightarrow$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \overline{w}} \quad \text{and} \quad \frac{\partial f}{\partial \overline{w}} = -\frac{\partial g}{\partial \bar{z}}.$$

□

Example 4.2. Let us illustrate Theorem 4.1 with the following examples:

(1) The function $\Phi(\xi) = {}^t\bar{\xi}$ satisfies for all $\xi = z + iw \in \mathbb{H}_{\mathbb{C}}$,

$$\Phi(z+iw) = {}^t\bar{z} - i\,{}^t\overline{w} = f(z,w) + ig(z,w).$$

We have $\frac{\partial f}{\partial \bar{z}} = Tr(\mathbb{1}) \neq \frac{\partial g}{\partial \overline{w}} = -Tr(\mathbb{1})$ and $\frac{\partial f}{\partial \overline{w}} = -\frac{\partial g}{\partial \bar{z}} = 0$. Thus, Φ is not holomorphic on $\mathbb{H}_{\mathbb{C}}$.

(2) The function $\Phi : \begin{array}{ccc} \mathbb{H}_{\mathbb{C}} & \rightarrow & \mathbb{C} \\ z+iw & \mapsto & \det(z+iw) \end{array}$ satisfies

$$\Phi(z+iw) = \det(z) - \det(w) + iTr\left(z^t\overline{w}\right) = f(z,w) + ig(z,w).$$

Since $f(z,w) = f(\bar{z},\overline{w})$ and $g(z,w) = g(\bar{z},\overline{w})$, then, an easy computation gives $\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \overline{w}} = Tr(z)$ and $\frac{\partial f}{\partial \overline{w}} = -\frac{\partial g}{\partial \bar{z}} = -Tr(w)$. Hence Φ is holomorphic on $\mathbb{H}_{\mathbb{C}}$.

(3) The function $\Phi : \begin{array}{ccc} \mathbb{H}_{\mathbb{C}} & \rightarrow & \mathbb{H}_{\mathbb{C}} \\ \xi & \mapsto & \xi^2 \end{array}$ is complex differentiable on $\mathbb{H}_{\mathbb{C}}$. Indeed, Φ can be decomposed into $\Phi = f + ig$ where $f(z, w) = z^2 - w^2$ and $g(z, w) = zw + wz$, $(z,w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$. In addition, an easy computation shows that $\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \overline{w}} = 0$ and $\frac{\partial f}{\partial \overline{w}} = -\frac{\partial g}{\partial \bar{z}} = 0$. Therefore, Φ is holomorphic on $\mathbb{H}_{\mathbb{C}}$.

(4) The function $\Phi : \begin{array}{ccc} \mathbb{H}_{\mathbb{C}} & \rightarrow & \mathbb{H}_{\mathbb{C}} \\ \xi & \mapsto & \bar{\xi}^2 \end{array}$ is complex differentiable only at $\xi = 0$. Moreover, Φ can be decomposed into $\Phi = f + ig$ such that $f(z,w) = \bar{z}^2 - \overline{w}^2$ and $g(z,w) = -\overline{zw} - \overline{wz}$, $(z,w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$. An easy computation yields

$$\frac{\partial f}{\partial \bar{z}} = 2\bar{z}, \quad \frac{\partial g}{\partial \overline{w}} = -2\bar{z}, \quad \frac{\partial f}{\partial \overline{w}} = -2\overline{w} \quad \text{and} \quad -\frac{\partial g}{\partial \bar{z}} = 2\overline{w}.$$

The equations $\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{w}}$ and $\frac{\partial f}{\partial \bar{w}} = -\frac{\partial g}{\partial \bar{z}}$ are simultaneously satisfied only at $\xi = (z, w) = (0_{\mathcal{M}_{\mathbb{H}}}, 0_{\mathcal{M}_{\mathbb{H}}})$. Hence Φ is not holomorphic.

The following result is a consequence of Theorem 4.1. It provides a principal example of complex quaternionic holomorphic function of one variable. Moreover, it shows that we have a lot of complex quaternionic holomorphic functions of one complex quaternion variable.

Theorem 4.3. If $S = \{\xi \in \mathbb{H}_{\mathbb{C}} : \det\xi = 0\}$ and $\Phi(\xi) = \xi^{-1}$ is the complex quaternionic inversion function defined for all $\xi \in \mathbb{H}_{\mathbb{C}} \setminus S$, then it holds that,

(i) *Φ has a decomposition into $f + ig$ where f and g are functions of two variables $(z, w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$ satisfying the Cauchy–Riemann type equations $\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{w}}$ and $\frac{\partial f}{\partial \bar{w}} = -\frac{\partial g}{\partial \bar{z}}$.*

(ii) *Φ is a biholomorphism from $\mathbb{H}_{\mathbb{C}} \setminus S$ to $\mathbb{H}_{\mathbb{C}} \setminus S$.*

Proof. First of all, it is clear that Φ is one-to-one since $\Phi \circ \Phi = Id_{\mathbb{H}_{\mathbb{C}} \setminus S}$. Furthermore, if $\xi = \begin{pmatrix} z_1 + iw_1 & z_2 + iw_2 \\ -\bar{z}_2 - i\bar{w}_2 & \bar{z}_1 + i\bar{w}_1 \end{pmatrix} \in \mathbb{H}_{\mathbb{C}} \setminus S$, then $\xi^{-1} = \Phi(\xi)$ can be written as follows

$$\begin{aligned}
\xi^{-1} &= \frac{1}{\det z - \det w + iTr\left(z^t\overline{w}\right)} \begin{pmatrix} \bar{z}_1 + i\bar{w}_1 & -z_2 - iw_2 \\ \bar{z}_2 + i\bar{w}_2 & z_1 + iw_1 \end{pmatrix} \\
&= \frac{1}{\det z - \det w + iTr\left(z^t\overline{w}\right)} \left[{}^t\bar{z} + i.{}^t\overline{w}\right] = f(z, w) + ig(z, w),
\end{aligned}$$

where f and g are functions of tow variable $(z, w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$ such that

$$\begin{aligned}
f(z, w) &= \frac{\det z - \det w}{(\det z - \det w)^2 + \left(Tr\left(z^t\overline{w}\right)\right)^2} . {}^t\bar{z} \\
&\quad + \frac{Tr\left(z^t\overline{w}\right)}{(\det z - \det w)^2 + \left(Tr\left(z^t\overline{w}\right)\right)^2} . {}^t\overline{w} \\
&= u\left(z, \bar{z}, w, \overline{w}\right). {}^t\bar{z} + v\left(z, \bar{z}, w, \overline{w}\right). {}^t\overline{w}
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
g(z, w) &= -\frac{Tr\left(z^t\overline{w}\right)}{(\det z - \det w)^2 + \left(Tr\left(z^t\overline{w}\right)\right)^2} . {}^t\bar{z} \\
&\quad + \frac{\det z - \det w}{(\det z - \det w)^2 + \left(Tr\left(z^t\overline{w}\right)\right)^2} . {}^t\overline{w} \\
&= -v\left(z, \bar{z}, w, \overline{w}\right). {}^t\bar{z} + u\left(z, \bar{z}, w, \overline{w}\right). {}^t\overline{w}.
\end{aligned} \tag{4.2}$$

Furthermore, the partial derivatives of f and g are such that

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}}(z,w) &= \frac{\partial u}{\partial \bar{z}}(z,w).{}^t\bar{z} + u(z,w).\mathbb{1} + \frac{\partial v}{\partial \bar{z}}(z,w).{}^t\overline{w} \\ \frac{\partial g}{\partial \overline{w}}(z,w) &= -\frac{\partial \bar{v}}{\partial w}(z,w).{}^t\bar{z} + u(z,w).\mathbb{1} + \frac{\partial u}{\partial \overline{w}}(z,w).{}^t\overline{w}. \end{aligned} \tag{4.3}$$

$$\begin{aligned} \frac{\partial f}{\partial \overline{w}}(z,w) &= \frac{\partial u}{\partial w}(z,w){}^t\bar{z} + v(z,w).\mathbb{1} + \frac{\partial v}{\partial w}(z,w).{}^t\overline{w} \\ -\frac{\partial g}{\partial \bar{z}}(z,w) &= \frac{\partial v}{\partial z}(z,w).{}^t\bar{z} + v(z,w).\mathbb{1} - \frac{\partial u}{\partial z}(z,w).{}^t\overline{w}. \end{aligned} \tag{4.4}$$

Since the equations $\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \overline{w}}$ and $\frac{\partial f}{\partial \overline{w}} = \frac{\partial g}{\partial \bar{z}}$ induces equalities of matrices of the form $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ in $\mathcal{M}_{\mathbb{H}}$, then following (4.1), (4.2), (4.3) and (4.4), f and g satisfy simultaneously the above Cauchy–Riemann equations, if and only if each of the followings holds

$$\begin{cases} \bar{z}_1\dfrac{\partial u}{\partial z} + \overline{w}_1\dfrac{\partial v}{\partial z} = -\bar{z}_1\dfrac{\partial v}{\partial w} + \overline{w}_1\dfrac{\partial u}{\partial w} \\ z_2\dfrac{\partial u}{\partial z} + w_2\dfrac{\partial v}{\partial z} = -z_2\dfrac{\partial v}{\partial w} + w_2\dfrac{\partial u}{\partial w}. \end{cases} \tag{4.5}$$

$$\begin{cases} \bar{z}_1\dfrac{\partial u}{\partial w} + \overline{w}_1\dfrac{\partial v}{\partial w} = \bar{z}_1\dfrac{\partial v}{\partial z} - \overline{w}_1\dfrac{\partial u}{\partial z} \\ z_2\dfrac{\partial u}{\partial w} + w_2\dfrac{\partial v}{\partial w} = z_2\dfrac{\partial v}{\partial z} - w_2\dfrac{\partial u}{\partial z}. \end{cases} \tag{4.6}$$

An easy computation of the partial derivatives $\frac{\partial u}{\partial z}$ and $\frac{\partial v}{\partial \overline{w}}$ yields

$$\frac{\partial u}{\partial \bar{z}}(z,w) = \frac{A(z,w) - B(z,w)}{C(z,w)} \quad \text{and} \quad \frac{\partial v}{\partial w}(z,w) = \frac{A(z,w) - B'(z,w)}{C(z,w)}$$

where

$$\begin{aligned} A &= \left[(\det z - \det w)^2 + \left(Tr\left(z^t\overline{w}\right)\right)^2\right].Tr(z) \\ B &= 2(\det z - \det w)\left[(\det z - \det w).Tr(z) + Tr\left(z^t\overline{w}\right).Tr(w)\right] \\ C &= \left[(\det z - \det w)^2 + \left(Tr\left(z^t\overline{w}\right)\right)^2\right]^2 \\ B' &= 2\left[(\det w - \det z).Tr(w) + Tr\left(z^t\overline{w}\right).Tr(z)\right]\left[Tr\left(z^t\overline{w}\right)\right]. \end{aligned}$$

Moreover, the partial derivatives

$$\frac{\partial u}{\partial \overline{w}}(z,w) \quad \text{and} \quad \frac{\partial v}{\partial \bar{z}}(z,w)$$

can be obtained directly using the facts that

$$u(\bar{z},\overline{w}) = -u(\overline{w},\bar{z}) \quad \text{and} \quad v(\bar{z},\overline{w}) = v(\overline{w},\bar{z}).$$

We get $\frac{\partial u}{\partial \overline{w}}(z,w) = -\frac{\partial u}{\partial \overline{z}}(w,z)$ and $\frac{\partial v}{\partial \overline{z}}(z,w) = \frac{\partial v}{\partial \overline{w}}(w,z)$. Since we have $\forall\ (z,w) \in \mathcal{M}_{\mathbb{H}} \times \mathcal{M}_{\mathbb{H}}$, $C(z,w) = C(w,z) > 0$, then formula (4.5) and (4.6) are equivalent to

$$\begin{cases} \bar{z}_1(2A - B - B')(z;w) = \overline{w}_1(B + B' - 2A)(w;z) \\ z_2(2A - B - B')(z;w) = w_2(B + B' - 2A)(w;z). \\ \bar{z}_1(B + B' - 2A)(w;z) = \overline{w}_1(B + B' - 2A)(z;w) \\ z_2(B + B' - 2A)(w;z) = w_2(B + B' - 2A)(z;w). \end{cases} \tag{4.7}$$

On the other hand, an easy computation provides immediately the equalities

$$(B + B')(z,w) = 2A(z,w) \quad \text{and} \quad (B + B')(w,z) = 2A(w,z).$$

Which permits to conclude that (4.7) are automatically satisfied and so that

$$\frac{\partial f}{\partial \bar{z}}(z,w) = \frac{\partial g}{\partial \overline{w}}(z,w) \quad \text{and} \quad \frac{\partial f}{\partial \overline{w}}(z,w) = -\frac{\partial g}{\partial \bar{z}}(z,w).$$

Hence, Φ and $\Phi^{-1} = \Phi$ both are holomorphic on $\mathbb{H}_{\mathbb{C}} \setminus S$ by Theorem 4.1. $\square$

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