Received 29 May 2023
Revised 20 June 2023
Accepted 20 June 2023

Optimal coalition splitting with heterogenous strategies

Raouf Boucekkine
Rennes School of Business France, Rennes, France
Carmen Camacho
PSE, Paris, France and
CNRS, Paris, France
Weihua Ruan
Purdue University Northwest, Hammond, Indiana, USA and
Centre for Unframed Thinking, Rennes School of Business, Rennes, France, and
Benteng Zou
University of Luxembourg, Luxembourg, Luxembourg

Abstract

Purpose – The authors characterize the conditions under which a country may eventually split and when it splits within an infinite horizon multi-stage differential game.

Design/methodology/approach – In contrast to the existing literature, the authors do not assume that after splitting, players will adopt Markovian strategies. Instead, the authors assume that while the splitting country plays Markovian, the remaining coalition remains committed to the collective control of pollution and plays open-loop.

Findings – Within a full linear-quadratic model, the authors characterize the optimal strategies. The authors later compare with the outcomes of the case where the splitting country and the remaining coalition play both Markovian. The authors highlight several interesting results in terms of the implications for long-term pollution levels and the duration of coalitions under heterogenous strategies as compared to Markovian behavior.

Originality/value – In this paper, the authors have illustrated the richness of the simplications of enlarging the set of strategies in terms of the emergence of coalitions, their duration and the implied welfare levels per player. Varying only three parameters (the technological gap, pollution damage and coalition payoff share distribution across players), the authors have been able to generate, among other findings, quite different rankings of welfare per player depending on whether the remaining coalitions after split play Markovian or stay precommited to the pre-splitting period decisions.

Keywords Coalition splitting, Environmental agreements, Differential games, Multistage optimal control, Precommitment vs Markovian

Paper type Research paper

1. Introduction

Is it a good idea to quit a coalition at all? If so, under which conditions and when? How shall the remaining members of the coalition react? Is it better for them to recompute their

JEL Classification — C61, C73, D71

This paper has been written in tribute to Ngo Van Long. The authors are very grateful to an anonymous referee for constructive comments. Of course the usual disclaimer applies. A preprint version of this paper, Boucekkine *et al.* (2022), appeared with the same title and has been posted as Discussion Paper of HAL SHS and DEM, respectively.

e-ISSN: 2635-0181
p-ISSN: 2635-0173
DOI 10.1108/FREP-05-2023-0021

strategies by themselves or to stick to the old commitments? The recent years have witnessed the split of international coalitions. Canada withdrew from Kyoto Protocol on December 13, 2011. In 2017, the U.S. ceased its participation in the 2015 Paris Agreement on climate change mitigation. More recently, the United Kingdom withdrew from the European Union on January 31, 2020. In all three examples, the remaining coalition did not change its strategy. Nor did so the CO_2 emission targets in the Kyoto or Paris agreements, or the trading rules within the EU. This behaviour of the remaining is utterly surprising. After a coalition collapses and that at least one of the old members recomputes her strategy, it is not optimal to remain loyal to old commitments. In this paper, we look into the "out of equilibrium" outcome in which one player re-optimizes its behaviour after the fall of a coalition but the remaining members don't.

Great attention has been paid to the emergence and stability of coalitions. The end of coalitions is receiving particular attention, from the study of the economic consequences of the Brexit (see Sampson, 2017 or the special issue of the Oxford Review of Economic Policy, vol 33, 2017 for example), to those of the U.S. withdrawal from the Kyoto Protocol and Paris agreement (e.g. Nong & Siriwardana, 2018; . . .). Ever since the 90s, great attention has been paid to the emergence and stability of coalitions, being specially active in a strand of the literature that blends environmental economics and game theory. The game-theoretical settings proposed have been quite diverse ranging from cooperative to non-cooperative, from static to dynamic through repeated games, and often including some interesting procedural ingredients, typically on enforceability of the agreements (see a survey in Bréchet, Gerard, and Tulkens (2011)).

In Boucekkine, Camacho, Ruan, and Zou (2022) we tackle the problem assuming that there exists a coalition which maximizes the joint payoff of all players taking into account the role of a public bad (or public good). Then, one player for a reason or another may decide to quit the coalition. There, we assume that both the quitting player and the remaining coalition would recompute their optimal trajectories after the split, that is we study the case where the players play Markovian after the split. In this paper, we continue our work in Boucekkine *et al.* (2022) and take another modelling perspective which serves a different research question. More precisely, we complete the study in Boucekkine *et al.* (2022) by analyzing the outcomes of an alternative game-theoretic configuration when the remaining coalition **does not modify its strategy**. In this case, the resulting trajectory after the split is not a subgame perfect Nash equilibrium, meaning that the remaining members of the coalition could most surely improve their payoff. Indeed, their strategy is stuck to the times in which the splitting player was part of the coalition and does not respond to her change of strategy. With the exception of Zou (2016) and to the best of our knowledge, there is no other contribution analyzing theoretically this strategic heterogeneity feature. We shall provide some preliminary insight in this paper. Beside the theoretical exploration of this case, we shall add some useful numerical exercises comparing the above heterogenous strategies case with the full Markovian setting studied in Boucekkine *et al.* (2022).

Our analysis builds on two realistic assumptions which depart from the typical dynamic analysis of coalition breakdown. One is borrowed from Boucekkine *et al.* (2022). We assume that when engaging into a coalition, players would in principle behave *as if* the coalition will last for ever. In some competing theories (for example in some related multistage games, see Carraro & Siniscalco, 1993), the possibility of splitting is given to players at a certain game stage, and this aspect is internalized by the players in establishing their strategies from the start of the coalitions. We do not follow this line which is in our view less realistic than our working assumption mentioned just above. As a result, our mathematical framework is markedly different. Last but not least, we explore in this paper the case where the remaining coalition remains committed to its initial coalition-like optimal behavior. Again, this corresponds to some of the most relevant recent cases, in particular the behavior of the

European Union after the Brexit or the behavior of the proponents of Kyoto Protocol (or Paris COP 21 conference) after some countries withdrew.

From the technical point of view, we consider a full linear-quadratic bimodal differential game setting, and we characterize the optimal trajectories before the split, and the strategies after the split. That is, we are able to provide with the entire sequence (starting with the initial cooperative game phase) and uncover the conditions under which splitting occurs at finite time. To this end, we make use of multistage optimal control techniques together with the typical techniques needed to solve differential games. Regarding multistage optimal control, an increasing number of papers characterize optimal/equilibrium regime transitions and the inherent timings building on the seminal exploration of Tomiyama (1985) (see for instance Boucekkine, Pommeret, & Prieur, 2013; Moser, Seidl, & Feichtinger, 2014; or Saglam, 2011). However, the literature is extremely thin when it comes to merging multi-stage optimal control and dynamic games (see for example Boucekkine, Krawczyk, & Vallée, 2011).

Section 4 carries out some numerical exercises to quantify the impact not only of quitting a coalition aimed at fighting a common bad, but to compare the effect of the strategic choice of the remaining player. What we do in practice is to compare the equilibrium trajectories we obtain in the present paper with the Nash Subgame perfect equilibrium in Boucekkine *et al.* (2022, 2022). In particular, we run four different exercises. In the first two, both the quitting player and the remaining coalition have the same technology, they are equally developed in economic terms. Our simulations show that when damage from the common bad is high, then the coalition lasts much less than when damage is moderate in case player J sticks to the initial commitment. Not only that, it is also shown that player i obtains a larger payoff when player J sticks to the initial commitment! This amounts to free-riding in the special theoretical context considered. We also study two scenarios in which player i has a more advanced technology. Generally speaking, we observe that when player i is technologically more advanced than J, then the coalition always lasts longer than in the Nash equilibrium. If player J will not play optimally after the coalition splits, then player i prefers to quit the coalition earlier (with respect to the Markovian case) to maximize their own individual payoff.

The paper is organized as follows. Section 2 briefly presents the differential game setting. Using a linear-quadratic game to analyze the coalition, Section 3 provides the optimal splitting time assuming that the splitting player plays Markovian and that the remaining player stays committed to the collective control of pollution and play *open-loop*. Section 4 compares numerically the optimal results of the present paper with those in Boucekkine *et al.* (2022, 2022), where both the splitting and the remaining player adopt Markovian strategies after the split. Section 5 concludes.

2. The model and the coalition regime

In this section we present our benchmark, first introduced in Boucekkine *et al.* (2022). For the readers' convenience, we develop all the steps of our modelling, even those that are identical to our previous work, and show the first best results obtained in Boucekkine *et al.* (2022).

2.1 The model

Suppose there exists a coalition, such as the Kyoto Protocol or the Paris Climate Agreement, and that one of the coalition's players, named player i, considers to quit the bloc at some future and endogenous date $T \geq 0$. The rest of the coalition is named player J. Note that T does not need to be finite. In this case, the coalition remains as a bloc forever.

When players act as one bloc, players i and J choose jointly the level of variables $x_i, x_J \in [0, X] \subset [0, +\infty)$, which are transformed into the final consumption good by players i and J respectively, and which provide them with joint utility. The choices for x_i and x_J increase the level of a stock variable y, which induces a loss in the joint utility. Let us assume that players play cooperatively until time T, when player i decides to quit the bloc. Note that at time T, player J may also decide to switch her strategy in response to i's change. Like in Boucekkine *et al.* (2022), we will illustrate our game by means of an environmental game. In this game, the coalition decides on the level of consumption of each agent, x_i and x_J, which will increase the stock of pollution, y.

We follow here the literature (see Dockner & Van Long, 1993; Dockner, Jorgensen, Van Long, & Sorger, 2000; Bertinelli, Camacho, & Zou, 2014; etc), and take linear-quadratic functional forms. Accordingly, the individual utility functions are

$$u_i(x_i) = a_i x_i - \frac{x_i^2}{2}, \quad u_J(x_J) = a_J x_J - \frac{x_J^2}{2},$$

and the pollution damage functions are

$$c_j(y) = \frac{by^2}{2}, \quad j = i, J.$$

In other words, regardless of the development level, the pollution damage is the same for both players for simplicity. x_j can be understood as the pollution emission of player j in order to produce final consumption goods, and a_j as the efficiency parameter which converts the pollution into the consumption good. Thus, a higher a_i indicates more advanced economies which can obtain more of the final consumption good emitting the same amount of pollution.

The objective of the players in the coalition is to maximize joint overall welfare, defined as

$$\max_{x_i, x_J} W(\infty) = \int_0^{+\infty} e^{-rt}[u_i(x_i) + u_J(x_J) - c_i(y) - c_J(y)]dt, \tag{1}$$

where r is the time discount rate. Decisions are subject to the dynamic constraint:

$$\dot{y}(t) = f(x_i, x_J, y) = x_i + x_J - \delta y(t), \quad y(0) = y_0 \text{ given}, \tag{2}$$

and $\delta \in [0, 1]$ is the natural reabsorption rate of CO_2 in the atmosphere.

Suppose that player i's share in total welfare is $\alpha \in (0, 1)$ and that the remaining share, $1 - \alpha$, belongs to the rest of the coalition. Then, the welfare of players i and J are

$$W_i = \alpha W(\infty) \text{ and } W_J = (1 - \alpha) W(\infty).$$

As in Boucekkine *et al.* (2022), the share α is independent of the amount x_i, it is given at the initial date and we assume that a renegotiation about the value α is impossible or too costly.

If player i quits the bloc at time T, then she obtains a share α of overall welfare until time T. From time T onwards, player i's objective becomes

$$W_{i,II} = \max_{x_i} \int_T^{+\infty} e^{-rt}[u_i(x_i) - c_i(y)]dt. \tag{3}$$

In Boucekkine *et al.* (2022) player J plays in Markovian strategies after the end of the coalition, and she faces the following problem after the split

$$W_{J,II} = \max_{x_J} \int_T^{+\infty} e^{-rt}[u_J(x_J) - c_J(y)]dt. \tag{4}$$

Here, we take a different approach and assume that player J will stay committed to the original trajectory, the one she decided together with player i. As we will discuss in the following section, this implies that the resulting equilibrium may not be any longer a Nash Subgame perfect equilibrium.

Since players are still bound by the common state variable y, their decisions are subject to the same state equation:

$$\dot{y}(t) = f(x_i, x_J, y) = x_i + x_J - \delta y(t), \quad t \geq T, \tag{5}$$

where the initial condition $y(T)$ comes from the outcome of the first period.

The optimal switching time for player i is obtained solving

$$\max_T \left(\alpha W(T) + \int_T^{+\infty} e^{-rt} [u_i(x_i) - c_i(y)] dt \right) = \max_T [\alpha W(T) + W_{i,II}], \tag{6}$$

where $W(T)$ is the same integral we obtained in (1), but over the finite time interval $[0, T]$, that is:

$$\int_0^T e^{-rt} [u_i(x_i) + u_J(x_J) - c_i(y) - c_J(y)] dt.$$

2.2 Joint optimal choice under coalition

The joint welfare optimization problem is

$$\max_{x_i, x_J} W(\infty) = \int_0^{+\infty} e^{-rt} \left(a_i x_i + a_J x_J - \frac{x_i^2 + x_J^2}{2} - b y^2 \right) dt, \tag{7}$$

subject to the following dynamic constraint:

$$\dot{y}(t) = x_i + x_J - \delta y(t), \quad y(0) = y_0 \ \text{ given}, \tag{8}$$

where $\delta \in [0, 1]$ is the depreciation rate.

Boucekkine *et al.* (2022) obtain the first best solution for $t \in [0, +\infty)$ for the coalition's optimal control problem above, and demonstrate that for any positive constants $b, r, \delta,$ and for any state trajectory $y(t)$, the optimal choices for players i and J are

$$x_j^*(y) = a_j + B + Cy, \quad j = i, J,$$

where

$$C = \frac{r + 2\delta - \sqrt{(r + 2\delta)^2 + 16b}}{4} (< 0), \quad \text{and} \quad B = \frac{(a_i + a_J)C}{r + \delta - 2C} (< 0).$$

For any $t \geq 0$ the optimal trajectory of the state variable y is

$$y(t) = \left(y_0 - y^* \right) e^{(2C - \delta)t} + y^*$$

where y^* is the optimal asymptotically stable long-run steady state and given by

$$y^* = \frac{a_i + a_J + 2B}{\delta - 2C} = \frac{r+\delta}{\delta(r+\delta) - 4b}(a_i + a_J)(>0). \tag{9}$$

Suppose player i quits the bloc at time T. Then, the above optimal choice continues until $t = T$ and pollution accumulation reaches

$$y(T) = \left(y_0 - y^*\right)e^{(2C-\delta)T} + y^* \tag{10}$$

where T is to be determined and it depends on the choice of the strategic space after the splitting.

The total payoff of player i just before the splitting is

$$\alpha W(T) = \alpha\left[\frac{\left(a_i^2 + a_J^2 - 2B^2\right)(1 - e^{-rT})}{2r} - 2BC\int_0^T e^{-rt}y(t)dt - \left(C^2 + b\right)\int_0^T e^{-rt}y^2(t)dt\right]. \tag{11}$$

It is straightforward to show that

$$\frac{dW(T)}{dT} = e^{-rT}\left[\frac{a_i^2 + a_J^2 - 2B^2}{2} - 2BCy(T) - \left(C^2 + b\right)y^2(T)\right] > 0 \tag{12}$$

if and only if

$$y(T) = \left(y_0 - y^*\right)e^{(2C-\delta)T} + y^* \in \left(0,\ \underline{y}\right)$$

where

$$\underline{y} = \frac{-2BC + \sqrt{4B^2C^2 + 2\left(C^2 + b\right)\left(a_i^2 + a_J^2 - 2B^2\right)}}{2\left(C^2 + b\right)}(>0). \tag{13}$$

To study the situation where splitting happens in finite time, we impose the following condition on the parameters:

Assumption 1. The model parameters ensure that the following inequality holds:

$$y_0 < \underline{y} < y^*.$$

3. The optimal time for separation

One of the possible reasons why player i quits the bloc could be that player i does not want to commit any more, rather she would prefer to be able to update her optimal efforts at any time depending on the pollution state y. Two outcomes are plausible. In the first, player i recalculates her Markovian trajectory while player J does not change her strategy and sticks to the commitment to the initially formed coalition. As already mentioned in the previous sections, in this new situation where player i adopts a Markovian strategy, player J's commitment is no longer optimal, and as a result, the outcome can not be a Nash equilibrium. We name this kind of choices as heterogenous strategies to emphasize the asymmetric choices between different players. Alternatively, player J could choose a new anticipating open-loop strategy which would takes into account player i's new Markovian choice. This kind of performance, where one player adopts Markovian strategy while the other one plays open-loop commitment but taking into account the viral player's potential updating strategy, forms a heterogenous Nash equilibrium following the definition of Zou (2016). Nevertheless, we focus on the first case in the rest of this section.

Suppose player i exits the bloc at time T and that player J stays with her original commitment to the coalition after T. Thus, the differential game is reduced after T to a standard optimal control problem for player i:

$$\max_{x_i} W_{II}^i \equiv \int_T^{+\infty} e^{-rt}\left(a_i x_i - \frac{x_i^2}{2} - \frac{b\,y^2}{2}\right)dt,$$

subject to

$$\dot{y} = x_i + x_J^* - \delta y, \quad \forall t \geq T,$$

with $y(T) = (y_0 - y^*)e^{(2C-\delta)T} + y^*$ and $x_J^*(y) = a_J + B + Cy$ given. The system is still autonomous and it is defined over an infinite time horizon. Proposition 1 below provides with the optimal trajectory of player i [1].

Proposition 1. Suppose that player i quits the coalition at time T, and that player J keeps her initial commitment. Then for any t ≥ T, the optimal Markovian strategy of player i is

$$x_i^i(y) = a_i + B^i + C^i y, \quad \forall y. \tag{14}$$

Furthermore, given the initial condition y(T), the corresponding state variable $y^i(t)$ is given by

$$y^i(t) = \left[y(T) - \hat{y}^i\right]e^{\left(C^i + C - \delta\right)(t-T)} + \hat{y}^i \quad \forall t \geq T,$$

where $\hat{y}^i$ is the asymptotically stable long-run steady state and it is given by

$$\hat{y}^i = \frac{a_i + a_J + B + B^i}{\delta - C - C^i},$$

and parameters

$$C^i = \frac{-2(C-\delta) - \sqrt{4(C-\delta)^2 + 4b(1-r)}}{2(1-r)} (< 0),$$

$$B^i = \frac{(a_i + a_J + B)C^i}{r + \delta - C - C^i} (< 0).$$

Obviously, since player J does not update her choice after the collapse of the coalition, the pair (x_J^*, x_i^i) may not be a Nash equilibrium given that x_J^* may not be the optimal response from player J's point of view, after the collapse of the coalition. Nonetheless, as discussed in Section 1, this is one possible choice among others. Boucekkine *et al.* (2022) investigate the situation where both players adopt Markovian strategies after the separation. The other possible choice of strategies are that both players recommit to play open-loop strategies, regardless of the collapse of the coalition. Of course, it may also happen that player i adopts Markovian strategy while player J plays adapted open-loop strategy by taking into account the Markovian choice of player i, form a Heterogeneous Nash equilibrium a la Zou (2016) and Dockner *et al.* (2000).

From the above proposition, it is straightforward to obtain the optimal value function W^i_{II}:

$$\begin{aligned}
W^i_{i,II} &= \int_T^{+\infty} e^{-rt}\left(a_i x_i - \frac{x_i^2}{2} - \frac{by^2}{2}\right)dt \\
&= \frac{a_i^2 - \left(B^i\right)^2}{2}\int_T^{+\infty} e^{-rt}dt - B^i C^i \int_T^{+\infty} e^{-rt} y^i(t)dt - \frac{\left(\left(C^i\right)^2 + b\right)}{2}\int_T^{+\infty} e^{-rt}\left(y^i\right)^2 dt \\
&= \left[\frac{a_i^2 - \left(B^i\right)^2}{2} - \widehat{y}^i B^i C^i - \frac{\left(C^i\right)^2 + b}{2}\left(\widehat{y}^i\right)^2\right]\frac{e^{-rT}}{r} \\
&+ \frac{e^{-rT}}{C^i + C - \delta - r}\left(B^i C^i\left[y(T) - \widehat{y}^i\right] + \left[\left(C^i\right)^2 + b\right]\widehat{y}^i\left[y(T) - \widehat{y}^i\right]\right) \\
&+ \frac{e^{-rT}}{2\left(C^i + C - \delta\right) - r}\frac{\left[\left(C^i\right)^2 + b\right]}{2}\left[y(T) - \widehat{y}^i\right]^2.
\end{aligned} \tag{15}$$

If $\frac{dW^i_{i,II}(T)}{dT} > 0$ for all T, then the coalition will last forever, that is, $T = \infty$. If the other polar case is true, $\frac{dW^i_{i,II}(T)}{dT} < 0$ for all T, then the coalition should stop immediately, that is $T = 0$. From the above expression, $\frac{dW^i_{i,II}}{dT}$ depends on $\frac{dy(T)}{dT}$, which is not known. In order to get rid of this unknown term, we employ the same calculations as in Appendix A in Boucekkine *et al.* (2022), which yield

$$\begin{aligned}
\frac{dW^i_{i,II}}{dT} &= e^{-rT}\left[\left(2C - \delta - \frac{r}{2}\right)C^i y^2(T) + \left(B^i(2C - r - \delta) + C^i(a_i + a_J + 2B)\right)y(T)\right] \\
&\quad + e^{-rT}\left[B^i(a_i + a_J + 2B) - rA^i_i\right]
\end{aligned} \tag{16}$$

where $rA^i_i = \frac{(a_i + B^i)^2}{2} + (a_J + B)\,B^i$. Thus, $\frac{dW^i_{i,II}}{dT}$ is a second degree polynomial in the pollution stock at the separation time $y(T)$, with positive leading term $\left(2C - \delta - \frac{r}{2}\right)C^i > 0$. Hence, since the coefficient of the second order term is positive, $\frac{dW^i_{i,II}}{dT} < 0$ cannot happen for all T, and the coalition will always, for all parameter sets, last for some positive amount of time.

It is easy to check that player i quits the coalition in finite time if and only if at the minimum value of the polynomial on the right hand side of (16)

$$y_{\min}(T) = -\frac{B^i(2C - r - \delta) + C^i(a_i + a_J + 2B)}{2\left(2C - \delta - \frac{r}{2}\right)C^i}$$

we have that

$$e^{+rT}\frac{dW^i_{i,II}(y_{\min}(T))}{dT} = \frac{3\left[B^i(2C-r-\delta)+C^i(a_i+a_J+2B)\right]^2}{4\left(2C-\delta-\frac{r}{2}\right)C^i} + \left[B^i(a_i+a_J+2B)-rA^i_i\right] < 0.$$

Furthermore, a sufficient condition of the duration of coalition is not zero is

$$\frac{dW^i_{i,II}(y_0)}{dT} > 0.$$

In the following we focus on the case where the duration of the coalition is finite but not zero (interior optimal switching time). In this case, the first order condition of player i' social welfare

$$\alpha\frac{dW(T)}{dT} + \frac{dW^i_{i,II}}{dT} = 0 \tag{17}$$

yields the following results.

Proposition 2. Let Assumption 1 hold and assume that the model's parameters satisfy $\frac{dW^i_{i,II}(y_{\min}(T))}{dT} < 0$. Suppose player i quits the bloc at time T and that from T onwards she adopts the optimal choice given in Proposition 1, while J stays with her initial first best commitment plan. Suppose the sharing parameter α checks

$$\max\left\{F^i\left(\frac{a_J}{a_i}, b\right),\ G^i(b)\right\} < \alpha < 1. \tag{18}$$

where functions $G(b)$ and $F\left(\frac{a_J}{a_i}, b\right)$ are defined as

$$G^i(b) \equiv \frac{C^i(2C-\delta-r/2)}{C^2+b}$$

and

$$F^i\left(\frac{a_J}{a_i}, b\right) \equiv \frac{1+\left[\frac{3\left(C^i\right)^2}{\left(r+\delta-3C^i\right)^2} - \frac{4CC^i}{\left(r+\delta-3C^i\right)(r+\delta-2C)}\right]\left(\frac{a_J}{a_i}+1\right)^2}{\left(\frac{a_J}{a_i}\right)^2 + 1 - \frac{2C^2}{(r+\delta-2C)^2}\left(\frac{a_J}{a_i}+1\right)^2}.$$

Furthermore, suppose that the pollution quantity at switching time T, given by

$$y^i_T = \frac{-\Sigma^i - \sqrt{\left(\Sigma^i\right)^2 - 4\Lambda^i\Gamma^i}}{2\Lambda^i} \tag{19}$$

satisfies

$$y_0 < y^i_T < y^*. \tag{20}$$

Then, player *i's* unique optimal quitting time $T^i \in (0, +\infty)$ is given by

$$T^i = \frac{1}{2C-\delta}\ln\left(\frac{y_T^i - y^*}{y_0 - y^*}\right), \tag{21}$$

where

$$\begin{cases} \Lambda^i = -\alpha\left(C^2 + b\right) + C^i\left(2C - \delta - \frac{r}{2}\right), \\ \Sigma^i = -2\alpha BC + B^i(2C - \delta - r) + C^i(a_i + a_J + 2B), \\ \Gamma^i = \frac{1}{2}\alpha\left(a_i^2 + a_J^2 - 2B^2\right) - rA_i^i + B^i(a_i + a_J + 2B). \end{cases} \tag{22}$$

The proof is similar to that shown in Appendix A in Boucekkine *et al.* (2022), so we omit the detailed calculations here. Fundamentally, given the linear-quadratic framework, the first order condition of optimal switching, equation (17), yields a second degree polynomial in term of pollution stock $y(T)$. The second order condition is easy to check if there is finite positive switching stock before the stock reaching long-run steady state y^*. Thus, the proposition is obtained by providing sufficient conditions under which the second degree polynomial has a unique positive root. Functions $G^i(b)$ and $F^i\left(\frac{a_J}{a_i}, b\right)$ come from the first order condition to guarantee the existence of positive and finite time such that the coalition collapses.

At first sight, both the separation condition (18) and the separation time (21) look like those in Proposition 3 in Boucekkine *et al.* (2022). This is not surprising, given that the only difference between the current work and Boucekkine *et al.* (2022) is the choice of the strategy space after the collapse of the coalition. The comparison unveils the importance of the strategy space choices, which matters not only for the condition of collapse but also the time when the separation happens. It is straightforward for given initial condition y_0 and first best potential long-run steady state y^*, duration of the coalition under different choice of strategies rely solely on the pollution level at separation, y_T^i. y_T^i is threshold level of pollution such that the first order condition (17) holds.

4. Some important implications of strategic heterogeneity: a numerical exploration

We would like to put in perspective our results by comparing the optimal splitting time obtained in the previous section with the optimal spliting time that results when both players adopt Markovian strategies after the split. On the one hand, we have T^i, the splitting time obtained here, when player J sticks to her initial commitment and does not recompute her optimal strategy. On the other hand stands T^m, the optimal splitting time where player J recomputes her optimal trajectory after the split and, as player i, adopts a Markovian strategy as studied in Boucekkine *et al.* (2022, 2022). Note that although the equilibrium reached in the first case may not be Nash subgame perfect, the resulting equilibrium in the second case is.

4.1 Calibration and scenarios

We present four different scenarios for players i and J and describe how the splitting times evolves with α, i's power in the coalition. In the first two, both players equally enjoy the final good, meaning that $a_i = a_J$ for two different damage parameters. In the last two scenarios, player i is technologically more advanced than player J to transform the stock variable into the final good, that is $a_i > a_J$. In these last exercises we shall be considering a slight technological gap and a large one.

Regarding the model calibration, note that both the models we are analysing rely solely on six parameters. Out of the six, we will let vary α, and we will use a_i, a_J and b to build the different scenarios. There are only r, the time discount rate, and δ, the depreciation rate, left. We fix $r = 0.015$, which is a value in line with Stern (2007). Indeed, this is an arguably low value, which makes complete sense in these models where players founded a coalition to take into account the evolution of the stock of pollution and limit the ensuing damage. δ, y's depreciation rate or its natural reabsorption rate takes the value 0.0005. This means that we are dealing here with a lasting pollutant, which constitutes a long-lived threaten to our players.

As mentioned, our simulations will compare the results obtained in this paper with those in Boucekkine *et al.* (2022, 2022). To ease the reading of this section, we provide next with the expressions of the most relevant variables when both players choose Markovian strategies after the split. In particular, the optimal splitting time T^m, the steady state value of the state variable, $\widehat{y}^m$, and functions $F^m\left(\frac{a_J}{a_i}, b\right)$ and $G^m(b)$:

$$T^m = \frac{1}{2C-\delta}\ln\left(\frac{y_T^m - y^*}{y_0 - y^*}\right),$$

$$y_T^m = \frac{-\Sigma - \sqrt{\Sigma^2 - 4\Lambda\Gamma}}{2\Lambda},$$

$$\widehat{y}^m = \frac{a_i + a_J + 2B^m}{\delta - 2C^m},$$

$$F^m\left(\frac{a_J}{a_i}, b\right) = \frac{1 + \left[\frac{3(C^m)^2}{(r+\delta-3C^m)^2} - \frac{4CC^m}{(r+\delta-3C^m)(r+\delta-2C)}\right]\left(\frac{a_J}{a_i}+1\right)^2}{\left(\frac{a_J}{a_i}\right)^2 + 1 - \left[\frac{2C^2}{(r+\delta-2C)^2}\right]\left(\frac{a_J}{a_i}+1\right)^2},$$

$$G^m(b) = \frac{C^m(2C-\delta-r/2)}{C^2+b},$$

where y_0 is the initial condition for y, y^* coincides with (9). Coefficients

$$C^m = \frac{(r+2\delta) - \sqrt{(r+2\delta)^2 + 12b}}{6}(<0), \quad B^m = \frac{(a_i+a_J)C^m}{r+\delta-3C^m}(<0).$$

and

$$\begin{cases} \Lambda^m = -\alpha\left(C^2+b\right) + C^m\left(2C - \delta - \frac{r}{2}\right), \\ \Sigma^m = -2\alpha BC + B^m(2C-\delta-r) + C^m(a_i + a_J + 2B), \\ \Gamma^m = \frac{1}{2}\alpha\left(a_i^2 + a_J^2 - 2B^2\right) - rA_i^m + B^m(a_i + a_J + 2B). \end{cases}$$

4.2 Findings

4.2.1 Long-term pollution levels. An important aspect to study is the implication of each strategy by the "remaining" coalition after the split for long-term pollution. Intuitively, precommitment would imply a lower pollution in the long-run compared to the case where the coalition plays Markovian after player i splits and play Markovian. What's definitely unclear is the size of the pollution saved thanks to precommitment in the long-run. Table 1 gives a comparison, across the four scenarios mentioned above, of the long-run levels of pollution, y^i (Resp. y^m) being the steady state pollution stock corresponding to the precommitment (Resp. Markovian) strategy.

Two important remarks can be made at this stage. First of all, somehow consistently with the literature comparing the long-term pollution outcomes in Markovian vs open-loop equilibria of standard two-player pollution games (see Dockner & Van Long, 1993, and the survey by Calvo & Rubio, 2013), it's quite clear that the precommitment of the "remaining" coalition brings pollution to long-term levels below those obtained if they would have played Markovian. Our context depart a lot from the standard pollution game setting but the comparison result is in the spirit of the related standard research.

Second, the pollution level is much lower in the long-run under precommitment of the coalition. Over the four scenarios considered in this paper, the steady state pollution level is consistently around 80% larger in the Markovian case! In terms of policy implications, this result is itself a quite relevant argument in favor of international cooperation to reduce global pollution. In our specific frame, it implies that if there were a world regulator entirely focusing on the pollution stock in the long-run, not only they should of course act against coalition splitting, but in addition to that, when splitting is unavoidable for any kind of (country-specific reasons), they should make sure that the remaining coalitions stays precommited. This seems the way several environmental and political coalitions have been working after the splitting of some of their major members in the last decades.

4.2.2 Splitting time and welfare- the case of homogenous coalitions. From now on, we will compare the splitting times and welfares across the scenarios described above. We shall pay attention to the role of coalition homogeneity in this respect, in particular in terms of technological level (parameters a_j, $j = i, J$). This aspect is key in the results shown in Boucekkine *et al.* (2022, 2022), it's therefore useful to perform out comparison through this lens.

The figures below show the optimal splitting time and the induced welfare for both players for the range of values for α that satisfies all feasibility assumptions in both this paper (Assumption 1, $\frac{dW^i_{i,II}(y_{\min}(T))}{dT} < 0$, (18) and (20)) when it regards T^i, and those in Boucekkine *et al.* (2022, 2022) regarding T^m so that comparisons can be made. In practice, this leads to consider only the α values that lead to finite time splitting to be optimal, that's by the latter conditions, for α large enough.

We start with the case of coalition homogeneity. Figure 1 shows the optimal splitting time and overall welfare of both players when $a_i = a_J$ for two different damage parameters. In particular, $a_i = a_J = 0.15$ and $b = 0.0001$ for the low damage case. In the higher damage case, we increase b by 50%, that is $b = 0.00015$. In Figure 1 the bold line represents the case when player J stays committed to the original strategy, and the dashed-dotted line the case when J plays Markovian. In the first example $\max\left\{F^i\left(\frac{a_J}{a_i}, b\right),\ G^i(b)\right\} > \max\left\{F^m\left(\frac{a_J}{a_i}, b\right),\ G^m(b)\right\}$.

	Case 1: $b = 0.0001$	Case 2: $b = 0.00015$	Case 3: $a_i = 0.2$	Case 4: $a_i = 0.175$
$\hat{y}^i$	15.1163	11.4002	17.6357	16.3760
$\hat{y}^m$	27.4713	19.7203	32.0498	29.7605

Source(s): Table by authors

Table 1. Steady state pollution levels for the 2 strategic cases

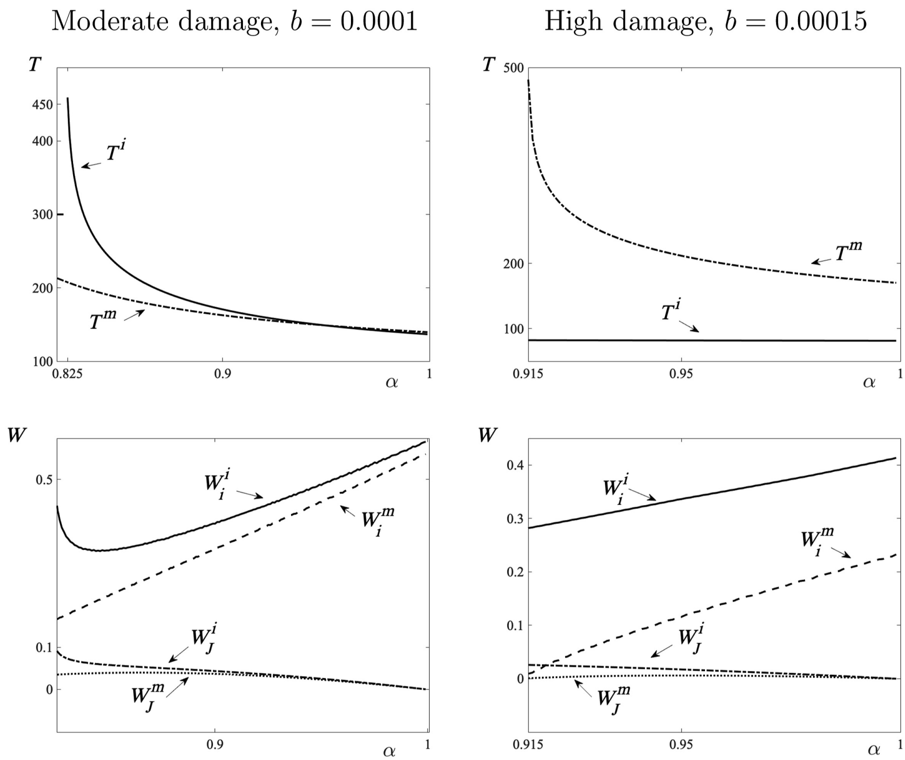

Figure 1.
Switching Times when $a_i = a_J = 0.15$. Top row: Optimal switching times

Note(s): Bottom row: Overall discounted welfare per player
Source(s): Figure by authors

Hence α needs to be larger in the first case, when J sticks to the original first best commitment than in the Markovian case.

Indeed, in all the numerical exercises we have conducted, this property holds true: interior coalition splitting requires α to be numerically much larger in the precommitment case. It is quite easy to figure out the rationale behind. Indeed, consistently with the full Markovian case studied in Boucekkine *et al.* (2022, 2022), player i would only enter and stay a certain time within a coalition (that's engaging in a finite time lived coalition) if the reward of being in the coalition, here the payoff share α, is large enough. If this player knows that the "remaining" coalition will play open-loop after they quit, they would be even less interested in joining the coalition and would prefer to free-ride obviously more than when the coalition recomputes their trajectories after the split. Hence, player i will ask for a larger α in the precommitment to dare joining such an environmental-friendly coalition. Numerically, we get this property for either homogenous or heterogenous coalitions (with player i more advanced technologically, see the case below) and whatever the pollution damage parameter.

When the damage cost is quite low, Figure 1 (left column) delivers a quite neat picture (for α values where finite time splitting optimally arises in the two strategic cases, here $\alpha = 0.82$). In this case, the duration of the coalition is always larger in the precommitment case. However, the difference drops sharply as α increases: while T^i more than doubles T^m around $\alpha = 0.82$, the two variables become very close when $\alpha > 0.9$. Indeed, as we shall see later, T^i may be lower than T^m for other parameterizations of the model. More importantly, when there is no technological advantage in joining the coalition (case of homogenous coalitions), the role of α becomes key in the creation and duration of coalitions. For the reasons made clear in the paragraph above, the latter has to be large enough for coalitions to be formed in all cases. However, as α goes up, the duration needed to extract "enough" payoff goes down. This is reflected clearly in Figure 1 (also in the right column with a much larger damage parameter).

What happens when damage increases is extremely interesting as shown on the right column of Figure 1. Several remarks are in order here. First of all, one should note that as the damage function is much larger than in the first scenario, and since there is no technological advantage for any player, the α values required to enter a coalition and to stay in for a while are higher than in the first scenario. Indeed, the player i does not enter the coalition until they make sure they retain around 91% of the coalition's payoff. Second, as in the case above, the duration of the coalitions decreases with α, sharply in the Markovian case, and much more slowly in the precommitment case. Third and related to the latter point, we get an ordering of coalition durations which is the reverse of the one obtained when the pollution damage is much lower. Even more, the coalition lasts significantly longer in the Markovian case for α close to 1 while the difference in duration becomes very small in the previous scenario (left column). When the pollution damage is increasingly important, the Markovian player finds it more profitable to stay in the coalition longer, especially at the quite high α values considered (larger than 0.9). The intuition behind this quite robust result is clear: for a given α large enough, and given that free-riding is by construction more important in the heterogenous startegy case, the splitting player will remain less in coalition in the latter case compared to the Markovian to take more advantage of the precommitment of the remaning coalition. This mechanism is entitled to be increasingly important as pollution cost rises. This is one example, among others, of the sensitivity of the orderings obtained in our comparison exercises to the values of the deep parameters of the model, here the pollution damage. Another important example is given in the next exercises where technological heterogeneity is incorporated into the story.

In the meanwhile, we also report for the information of the readers, the outcomes in terms of welfare, depending on the strategy and the value of the damage parameter (without technological heterogeneity). Not surprisingly, one can observe that either in the left or right column, player i's welfare is always larger than J's as a consequence of the precommitment of the latter player. Another intuitive outcome is that player i's welfare increases with α. The findings on the welfare of the "remaining" coalition, in particular on the comparison between the precommitment vs Markovian cases, are not particularly conclusive, but this might be due to the range of the α values required by the comparison exercise.

4.2.3 Splitting time and welfare - the case of heterogenous coalitions. Let us now consider technological heterogeneity, which is a very important ingredient theoretically and empirically. We fix $b = 0.0001$ and first explore the situation where player i has a better technology. In particular, we compare the case in which $a_i = 0.2$ and $a_J = 0.15$, that is, i's technology is a 33% more performant than J's. In the second heterogeneity case explored, we keep a_J constant and decrease a_i to 0.175, that is, the difference is only half of the initial case.

In both examples and similarly to the homogenous coalition case, we find that player i requires less of the coalition payoff share to split under Markovian strategies. We have already explained the rationale behind before. More interesting, player i remains longer in the coalition in the latter case (see Figure 2, both columns). This is also consistent with the outcomes of Figure 1, right column, when pollution damage is large, but this is not the pollution damage value in the technological heterogeneity scenarios here considered. As we have already pointed out, when the pollution damage is increasingly important, the Markovian player finds it more profitable to stay in the coalition longer, especially at the quite high α values considered (for the sake of comparison of coalition splitting in the two strategic

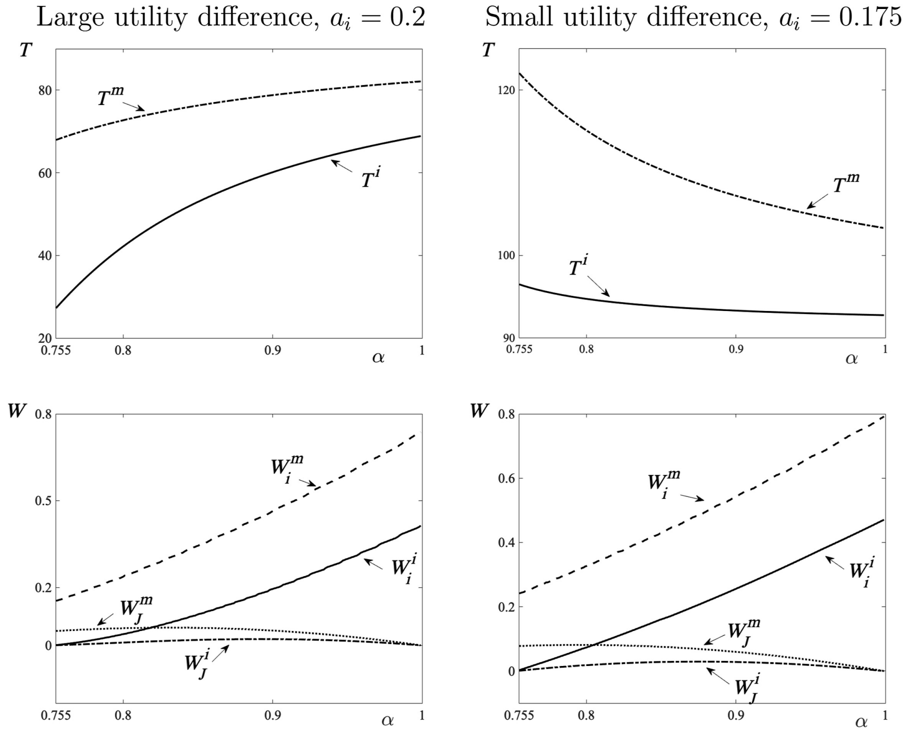

Figure 2.
Switching Times when $a_i > a_J$. Top row: Optimal switching times

Note(s): Bottom row: Overall discounted welfare per player
Source(s): Figure by authors

cases). If the pollution damage is low, then there is much less difference between the coalition durations in the two strategic cases as α rises. Figure 1, left column, illustrates this property as the two curves for T^i and T^m get very close as α goes up. Things are different if player i is far more advanced technologically as reflected in Figure 2, left column.

There are however more important differences with the homogenous coalition case studied before. One obvious and striking difference shows up in the time splitting values in Figure 2 and those in the left column of Figure 1 (same b value). Depending on the value of α, splitting times can be up to 6 or 7 times shorter, whatever the strategy followed by the coalition after splitting, when player i is technologically largely advanced. This is quite easy to grasp: as outlined in Boucekkine *et al.* (2022, 2022), a technologically advanced country has less incentives to enter and to remain in a coalition *ceteris paribus* where they somehow share their technology and their production capacity. Therefore, for given α checking the splitting condition, the larger the technological lead, the lower the duration of the coalition.

Another difference due to technological heterogeneity is worth mentioning: comparing Figure 1 with the left column of Figure 2, one can observe that while the splitting times are decreasing with α in the former, they are increasing in the latter. This is again quite easy to interpret. As mentioned just above, a technologically advanced country (with respect to the rest of the coalition) is typically reluctant to share its advanced technology, which is inherent in coalition formation. In this particular case, the main reason driving coalition formation is the value of α: Figure 2, left column, illustrates the case that large enough technological advance would lead the technological leader to stay longer (at the pollution damage level considered). When the technological advantage is not that big, which is the case of the right column of Figure 2, α will no longer drive the splitting behavior of the player i, and we recover the decreasing shapes of Figure 1 (under homogenous technology).

A last striking difference is the reversing in the order of welfare level for each player as we introduce technological heterogeneity. In Figure 1, for the two players, welfare in the precommitment case is larger than in the Markovian. It's the opposite with technological heterogeneity in the two scenarios considered. At this point, we are unable to bring out any conclusion on whether this numerical finding is robust enough. Depending on parameters, the individual welfares rank differently, which uncovers potentially interesting conceptual issues.

To conclude, we can put our exercises in perspective. Note that when both players are technically identical but i controls a larger part of coalition payoff than J, and in particular for large enough α, the coalition lasts longer in the precommitment case than in the Markovian. Furthermore, player i obtains a larger welfare! This is clearly the result of free-riding. When the technological gap widens, the coalition always lasts less in the precommitment case. Knowing that player J will not play optimally after the split, player i needs to quit the coalition earlier to maximize own individual payoff. Despite that, player i cannot make up for J's deviation.

5. Conclusion

The main purpose of this paper is to illustrate the importance of the strategy space choices in differential games in the context of a bimodal model, a coalition mode and a post-coalition mode. Our exercise builds on previous work by Boucekkine *et al.* (2022) that only considers the case where players act in a Markovian way after splitting. Whether the remaining coalition would necessarily play Markovian may be academically more relevant (for example, for the investigation of certain types of equilibria like Markov Perfect Equilibria) but it's not at all granted that getting out from the standard academic choices is less interesting even from the theoretical viewpoint. Also it goes without saying that if any, the so far observed evolutions of post-splitting episodes rather document cases of precommitment on the side of the remaining coalitions.

In this paper, we have illustrated the richness of the implications of enlarging the set of strategies in terms of the emergence of coalitions, their duration and the implied welfare levels per player. Varying only three parameters (the technological gap, pollution damage and coalition payoff share distribution across players), we have been able to generate, among other findings, quite different rankings of welfare per player depending on whether the remaining coalitions after split play Markovian or stay precommited to the pre-splitting period decisions. Several of these results deserve a more careful theoretical examination. This is clearly outside the scope of this paper.

Note

1. All proofs can be found in the Appendix.

References

Bertinelli, L., Camacho, C., & Zou, B. (2014). Carbon capture and storage and transboundary pollution: A differential game approach. *European Journal of Operational Research*, *237*, 721–728.

Boucekkine, R., Krawczyk, J., & Vallée, T. (2011). Environmental quality versus economic performance: A dynamic game approach. *Optimal Control Applications and Methods*, *32*, 29–46.

Boucekkine, R., Pommeret, A., & Prieur, F. (2013). Optimal regime switching and threshold effects. *Journal of Economic Dynamics and Control*, *37*, 2979–2997.

Boucekkine, R., Camacho, C., Ruan, W., & Zou, B. (2022). Why and when coalitions split? An alternative analytical approach with an application to environmental agreements. *working paper*. *HAL SHS Sciences humanes et Socials*, and DEM Working Paper, University of Luxembourg, DP2022-05. Available from: https://shs.hal.science/halshs-03676670

Boucekkine, R., Camacho, C., Ruan, W., & Zou, B. (2022). Optimal coalition splitting with heterogenous strategies. Working Paper. *HAL SHS Sciences humanes et Socials*, and DEM working paper, University of Luxembourg, DP2022-13. Available from: https://shs.hal.science/halshs-03770401

Bréchet, T., Gerard, F., & Tulkens, H. (2011). Efficiency vs. stability in climate coalitions: A conceptual and computational appraisal. *The Energy Journal*, *32*, 49–75.

Calvo, E., & Rubio, S. S. (2013). Dynamic models of international environmental agreements: A differential game approach. *International Review of Environmental and Resource Economics*, *6*, 289–339.

Carraro, C., & Siniscalco, D. (1993). Strategies for the international protection of the environment. *Journal of Public Economics*, *52*, 309–328.

Dockner, E., & Van Long, N. (1993). International pollution control: Cooperative versus noncooperative strategies. *Journal of Environmental Economics and Management*, *24*, 13–29.

Dockner, E., Jorgensen, S., Van Long, N., & Sorger, G. (2000). *Differential games in economics and management*. Cambridge: Cambridge University Press.

Moser, E., Seidl, A., & Feichtinger, G. (2014). History-dependence in production-pollution-trade-off models: A multi-stage approach. *Annals of Operations Research*, *222*, 457–481.

Nong, D., & Siriwardana, M. (2018). Effects on the U.S. Economy of its proposed withdrawal from the Paris agreement: A quantitative assessment. *Energy*, *159*, 621–629.

Saglam, C. (2011). Optimal pattern of technology adoptions under embodiment: A multi-stage optimal control approach. *Optimal Control Applications and Methods*, *32*, 574–586.

Sampson, T. (2017). Brexit: The economics of international disintegration. *Journal of Economic Perspectives*, *31*(4), 163–184.

Stern, N. (2007). *The economics of climate change: The stern review*. Cambridge University Press.

Tomiyama, K. (1985). Two-stage optimal control problems and optimality conditions. *Journal of Economic Dynamics and Control*, *9*, 317–337.

Zou, B. (2016). Differential games with (a)symmetric payers and heterogeneous strategies. *Journal of Reviews on Global Economics*, *5*, 171–179.

Appendix A

A.1 Proof of Proposition 1

Define the Bellman Value function as $V_i(y)$, which must check the following Hamilton-Jacobi-Bellman (HJB) equation for $t \geq T$,

$$rV_i(y) = \max_{x_i}\left[a_i x_i - \frac{x_i^2}{2} - \frac{b\,y^2}{2} + V_i'(y)\ \left(x_i + x_J^* - \delta y\right)\right],$$

where x_J^* is given by

$$x_J^* = a_J + B + Cy.$$

Given the linear-quadratic forms of the objective function and linear state equation, we can guess that the Bellman value function has an affine-quadratic form as:

$$V_i(y) = A_i + B_i y + \frac{C_i^2}{2} y^2,$$

Taking first order condition on the right hand side of the HJB equation, it yields the optimal choices

$$x_i(y) = a_i + V_i'(y) = a_i + B_i + C_i y, \quad \forall t \geq T.$$

Substituting the optimal choice into the right hand side of the HJB equation, $x_i(y)$ obtains

$$\begin{aligned}
RHS &= a_i(a_i + B_i + C_i y) - \frac{(a_i + B_i + C_i y)^2}{2} - \frac{by^2}{2} \\
&+(B_i + C_i y)[(a_i + B_i + C_i y) + (a_J + B + Cy) - \delta y] \\
&= a_i(a_i + B_i) + a_i C_i y - \frac{1}{2}\left[(a_i + B_i)^2 + 2C_i(a_i + B_i)y + C_i^2 y^2\right] - \frac{by^2}{2} \\
&+(B_i + C_i y)[(a_i + a_J + B_i + B) + (C_i + C - \delta)y] \\
&= a_i(a_i + B_i) - \frac{(a_i + B_i)^2}{2} + B_i(a_i + a_J + B_i + B) \\
&+[a_i C_i - C_i(a_i + B_i) + C_i(a_i + a_J + B_i + B) + B_i(C_i + C - \delta)]y \\
&+\left[-\frac{C_i^2 + b}{2} + C_i(C_i + C - \delta)\right]y^2 \\
&= \frac{a_i^2 - B_i^2}{2} + B_i^2 + B_i(a_i + a_J + B) + [C_i(a_i + a_J + B) + B_i(C_i + C - \delta)]y \\
&+\left[-\frac{C_i^2 + b}{2} + C_i(C_i + C - \delta)\right]y^2.
\end{aligned}$$

Comparing coefficients on both sides of the HJB equation, it follows

$$\begin{cases} rA_i = \dfrac{(a_i + B_i)^2}{2} + (a_J + B)\, B_i, \\ (r + \delta - C - C_i)B_i = (a_i + a_J + B)\, C_i, \\ (r + 2\delta)C_i = C_i^2 + 2C\, C_i - b. \end{cases} \tag{23}$$

Let us focus on the last equation.

$$(r + 2\delta)C_i = C_i^2 + 2C\, C_i - b \Rightarrow r\frac{C_i^2}{2} = \frac{-C_i^2 - b}{2} + C_i(C_i + C - \delta),$$

that is

$$(1 - r)C_i^2 + 2C_i(C - \delta) - b = 0,$$

which has two roots:

$$C_i = \frac{-2(C-\delta) \pm \sqrt{4(C-\delta)^2 + 4b(1-r)}}{2(1-r)}.$$

Given $C < 0$ and $\delta > 0$, and if $r < 1$, there is one positive and one negative root for C_i. Given that the Bellman value function must be concave, we can only take the negative value for C_i, that is

$$C_i = \frac{-2(C-\delta) - \sqrt{4(C-\delta)^2 + 4b(1-r)}}{2(1-r)}.$$

Furthermore,

$$B_i = \frac{(a_i + a_J + B)C_i}{r + \delta - C - C_i} (< 0).$$

Substituting the above optimal choice of player i and x_J^* into the state equation, it follows

$$\dot{y} = (a_i + a_J + B + B_i) + (C_i + C - \delta)y \quad \forall t \geq T$$

where $y(T)$ is unknown but will be determined after transversality condition is made. The solution is straightforward.

Corresponding author
Carmen Camacho can be contacted at: carmen.camacho@psemail.eu